CONNECTIONS ON PRINCIPAL PROLONGATIONS OF PRINCIPAL BUNDLES

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Abstract. We study the principal connections of the r-th principal prolongation W^rP of a principal bundle $P(M,G)$ by using the related Lie algebroids. We deduce that both basic approaches to the concept of torsion are naturally equivalent. We prove that the torsion-free connections on W^rP are in bijection with the reductions of $W^{r+1}P$ to the group $G^1_m \times G$. Special attention is paid to the flow prolongation of connections.

Consider a principal bundle $P(M,G)$, $\dim M = m$. Its r-th order principal prolongation W^rP is the bundle of all r-jets $j^r_{(0,e)}\varphi$ of local principal bundle isomorphisms

$$\varphi\colon \mathbb{R}^m \times G \to P\,, \quad 0 \in \mathbb{R}^m,\ e = \text{the unit of } G.$$

This is a principal bundle over M with structure group $W^r_mG := W^r_0(\mathbb{R}^m \times G)$, whose action on W^rP is given by the jet composition, [11]. If $G = \{e\}$ is the one-element group, then $M \times \{e\}$ is identified with M and $W^r(M \times \{e\}) = P^rM$ is the r-th order frame bundle of M. The r-th principal prolongation W^rP is a fundamental structure for both the general theory of geometric object fields, [11], and the gauge theories of mathematical physics, [3].

Our main subject are the principal connections on W^rP. So we omit the adjective "principal" as a rule. At a few places (in particular at the beginning of Section 4), where we mention arbitrary connections on an arbitrary fibered manifold Y, we call them explicitly "general connections".

It has been clarified recently that the connections Λ on P^rM are in bijection with the r-th order linear connections $\lambda\colon TM \to J^rTM$ on TM. In Section 2 we point out that in the case of W^rP the role of TM is replaced by the Lie algebroid $LP = TP/G$ of P. Using the flow prolongation of right -invariant vector fields, we identify $J^r(LP \to$

2000 *Mathematics Subject Classification*: 53A05, 58A20, 58A32.

Key words and phrases: principal prolongation of principal bundle, gauge theories, connection, torsion, Lie algebroid.

The author was supported by the Ministry of Education of the Czech Republic under the project MSM 0021622409 and the grant GACR No. 201/05/0523.

M) with the Lie algebroid LW^rP and prove that the connections Δ on W^rP are in bijection with the linear splittings $\delta\colon TM \to J^rLP$. The torsion of Δ is defined as the covariant exterior differential of the canonical one-form of W^rP, while the torsion of δ is introduced by means of the bracket of LP. In Section 3 we prove that the torsions of Δ and δ are naturally equivalent.

A connection Γ on P and a connection Λ on P^rM determine a connection $\mathcal{W}^r(\Gamma,\Lambda)$ on W^rP by means of the flow prolongation of vector fields. To demonstrate the applicability of the algebroid approach, we express explicitly the torsion of $\mathcal{W}^1(\Gamma,\Lambda)$ in terms of the torsion of Λ and the curvature of Γ in Section 4. For arbitrary r, we then deduce that $\mathcal{W}^r(\Gamma,\Lambda)$ is torsion-free, if and only if Λ is torsion-free and Γ is curvature-free. In Section 5 we prove that, analogously to the case of P^rM, the torsion-free connections on W^rP are identified with certain reductions of $W^{r+1}P$.

From a general point of view, J^r is a fiber product preserving bundle functor. In Section 6 we study an arbitrary functor F of this type and we deduce the algebroid formula for the flow prolongation of the above-mentioned pair of connections Γ and Λ.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [11].

1. The torsion of connections on W^rP. We write G^r_m for the r-th jet group in dimension m and $T^r_mG = J^r_0(\mathbb{R}^m, G)$. We have $W^rP = P^rM \times_M J^rP$ and

$$W^r_0(\mathbb{R}^m \times G) = W^r_mG = G^r_m \rtimes T^r_mG \tag{1}$$

is the group semidirect product with the group composition

$$(g_1, C_1)(g_2, C_2) = \big(g_1 \circ g_2, (C_1 \circ g_2) \bullet C_2\big)\,, \tag{2}$$

where $\bullet$ denotes the induced group composition in T^r_mG, [11]. The first product projection $W^rP \to P^rM$ is a principal bundle morphism with the associated group homomorphism $W^r_mG \to G^r_m$ determined by (1). Write $\mathcal{PB}_m(G)$ for the category of principal G-bundles with m-dimensional bases and principal G-morphisms with local diffeomorphisms as base maps. Let $\bar P(\bar M, G)$ be another object of $\mathcal{PB}_m(G)$ For every $\mathcal{PB}_m(G)$-morphism $f\colon P \to \bar P$ with base map $\underline{f}\colon M \to \bar M$, we define

$$W^rf = P^r\underline{f} \times_{\underline{f}} J^rf\colon P^rM \times_M J^rP \to P^r\bar M \times_{\bar M} J^r\bar P\,. \tag{3}$$

Then W^r is a functor from the category $\mathcal{PB}_m(G)$ into $\mathcal{PB}_m(W^r_mG)$, [11].

On P^rM, we have the canonical one-form $\varphi_r\colon TP^rM \to \mathbb{R}^m \times \mathfrak{g}_m^{r-1}$. On W^rP, we introduce analogously a canonical one-form $\Theta_r\colon TW^rP \to \mathbb{R}^m \times \mathfrak{w}_m^{r-1}G = T_{(0,e_{r-1})}W^{r-1}(\mathbb{R}^m \times G)$, e_{r-1} = the unit of $W_m^{r-1}G$. Consider $u = j^r_{(0,e)}\varphi \in W^rP$ and write $u_1 = \pi^r_{r-1}(u) \in W^{r-1}P$, where π^r_{r-1} is the jet projection. The tangent map

$$\widetilde{u} = T_{(0,e_{r-1})}W^{r-1}\varphi\colon \mathbb{R}^m \times \mathfrak{w}_m^{r-1}G \to T_{u_1}W^{r-1}P \tag{4}$$

is a linear isomorphism depending on u only. For every $Z \in T_uW^rP$, we define

$$\Theta_r(Z) = \widetilde{u}^{-1}\big(T\pi^r_{r-1}(Z)\big)\,.$$

Clearly, the following diagram commutes

$$\begin{array}{ccc} TW^rP & \xrightarrow{\ \Theta_r\ } & \mathbb{R}^m \times \mathfrak{w}_m^{r-1}G \\ \downarrow & & \downarrow \\ TP^rM & \xrightarrow{\ \varphi_r\ } & \mathbb{R}^m \times \mathfrak{g}_m^{r-1} \end{array} \tag{5}$$

For a connection Λ on P^rM, Yuen defined its torsion to be the covariant exterior differential $D_\Lambda\varphi_r$. Analogously, in [13] we introduced

Definition 1. The torsion of a connection Δ on W^rP is the covariant exterior differential $D_\Delta\Theta_r$.

2. Another approach to connections on W^rP. A linear splitting $TM \to J^rTM$ is said to be a linear r-th order connection on TM. Since P^r is an r-th order bundle functor from the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms into $\mathcal{PB}_m(G^r_m)$, the flow prolongation $\mathcal{P}^rX$ of every vector field X on M is a right-invariant vector field on P^rM, [11]. The restriction $\mathcal{P}^rX \mid P^r_xM$ depends on j^r_xX only. This defines an identification

$$I^r_M\colon J^rTM \to LP^rM\,, \tag{6}$$

where LP^rM is the Lie algebroid of P^rM, [15]. Clearly, for every linear splitting $\lambda\colon TM \to J^rTM$, the rule

$$\Lambda(Z) = I^r_M\big(\lambda(Z)\big)\,, \qquad Z \in TM \tag{7}$$

defines a connection Λ on P^rM. This establishes a bijection $\lambda \to \Lambda$ between linear r-th order connections on TM and connections on P^rM, [8].

Such a bijection can be generalized to the case of W^rP. The role of TM is replaced by the Lie algebroid $LP = TP/G$ of P. Every section $\sigma\colon M \to LP$ is identified with a right-invariant vector field $\bar\sigma\colon P \to TP$ such that $\sigma = \bar\sigma/G$. Since W^r is a functor from $\mathcal{PB}_m(G)$

into $\mathcal{PB}_m(W^r_mG)$, the flow prolongation $\mathcal{W}^r(\bar\sigma)$ is a right-invariant vector field on $\mathcal{W}^rP$. The rule

$$I^r_P(j^r_x\sigma) = \big(\mathcal{W}^r(\bar\sigma) \mid W^r_xP\big)/W^r_mG \tag{8}$$

defines a bijection $I^r_P\colon J^r(LP \to M) \to L\mathcal{W}^rP$. This is the principal bundle form of an identification that was established in the Lie algebroid form in [14]. In the same way as in (7), we obtain

Proposition 1. (8) *identifies connections* Δ *on* W^rP *with linear splittings*

$$\delta\colon TM \to J^rLP\,. \tag{9}$$

We say that δ *is the algebroid form of* Δ.

3. The torsions on W^rP and J^rLP. The $(r-1)$-jet at $x \in M$ of the bracket $[X_1, X_2]$ of two vector fields X_1 and X_2 on M depends on the r-jets $j^r_xX_1$ and $j^r_xX_2$. This defines a bilinear morphism

$$[\ ,\ ]_r\colon J^rTM \times_M J^rTM \to J^{r-1}TM\,.$$

For a linear r-th order connection $\lambda\colon TM \to J^rTM$, one defines its torsion $\tau\lambda\colon TM \times_M TM \to J^{r-1}TM$ by

$$(\tau\lambda)(Z_1, Z_2) = \big[\lambda(Z_1), \lambda(Z_2)\big]_r\,, \qquad (Z_1, Z_2) \in TM \times_M TM\,. \tag{10}$$

Our result from [8] reads: If Λ is the principal bundle form of λ, then the torsion $D_\Lambda\varphi_r$ is naturally identified with the torsion $\tau\lambda$. We are going to deduce the same result for the case of W^rP.

Since the bracket $[\![\ ,\ ]\!]$ of LP is a first order operator, it determines a bilinear morphism

$$[\![\ ,\ ]\!]_r\colon J^rLP \times_M J^rLP \to J^{r-1}LP$$

analogously to $[\ ,\ ]_r$.

Definition 2. The torsion of a connection in the algebroid form $\delta\colon TM \to J^rLP$ is the morphism

$$\tau\delta\colon TM \times_M TM \to J^{r-1}LP$$

defined by

$$(\tau\delta)(Z_1, Z_2) = [\![\delta(Z_1), \delta(Z_2)]\!]_r\,, \qquad (Z_1, Z_2) \in TM \times_M TM\,.$$

Clearly, $\tau\delta$ can be viewed as a section of $J^{r-1}LP \otimes \Lambda^2T^*M$.

Write $U^{r-1}_m = J^{r-1}_0L(\mathbb{R}^m \times G)$. Since $J^{r-1}L$ is an r-th order gauge natural bundle, every $u = j^r_{(0,e)}\varphi \in W^rP$ can be interpreted as a map

$$J^{r-1}L(u) := (J^{r-1}L)(\varphi)\mid_0\colon U^{r-1}_m \to J^{r-1}_xLP\,. \tag{11}$$

Our identification $I_P^{r-1}\colon J^{r-1}LP \to LW^{r-1}P$ is a natural equivalence of functors $J^{r-1}L$ and LW^{r-1}. Write $V_m^{r-1} = L_0W^{r-1}(\mathbb{R}^m \times G) = \mathbb{R}^m \times \mathfrak{w}_m^{r-1}$. Analogously to (11), we construct

$$LW^{r-1}(u) := LW^{r-1}(\varphi)\,|_0\colon V_m^{r-1} \to L_xW^{r-1}P\,. \tag{12}$$

The restriction of $I^{r-1}_{\mathbb{R}^m\times G}$ over $0 \in \mathbb{R}^m$ yields a bijection $\varepsilon\colon U_m^{r-1} \to V_m^{r-1}$. By naturality, the following diagram commutes

$$\begin{array}{ccc} U_m^{r-1} & \xrightarrow{J^{r-1}L(u)} & J_x^{r-1}LP \\ \Big\downarrow{\scriptstyle\varepsilon} & & \Big\downarrow{\scriptstyle (I_P^{r-1})_x} \\ V_m^{r-1} & \xrightarrow{LW^{r-1}(u)} & L_xW^{r-1}P \end{array} \tag{13}$$

If we identify $L_xW^{r-1}P$ with $T_{u_1}W^{r-1}P$, $u_1 = \pi^r_{r-1}(u)$, then $LW^{r-1}(u)$ is identified with $\widetilde{u}$ from (4). Since $J^{r-1}LP \otimes \Lambda^2T^*M$ is a fiber bundle associated to W^rP with standard fiber $U_m^{r-1} \otimes \Lambda^2\mathbb{R}^{m*}$ and $\tau\delta$ is a section, we can consider its frame form

$$\{\tau\delta\}\colon W^rP \to U_m^{r-1} \otimes \Lambda^2\mathbb{R}^{m*}\,, \tag{14}$$

[11]. On the other hand, $D_\Delta\Theta_r$ is a horizontal 2-form, so that it can be interpreted as a map

$$\{D_\Delta\Theta_r\}\colon W^rP \to V_m^{r-1} \otimes \Lambda^2\mathbb{R}^{m*}\,. \tag{15}$$

Further we can construct

$$\varepsilon \otimes \mathrm{id}_{\Lambda^2\mathbb{R}^{m*}}\colon U_m^{r-1} \otimes \Lambda^2\mathbb{R}^{m*} \to V_m^{r-1} \otimes \Lambda^2\mathbb{R}^{m*}\,.$$

Proposition 2. *Under the identifications* (14) *and* (15)*,*

$$\{D_\Delta\Theta_r\} = \big(\varepsilon \otimes \mathrm{id}_{\Lambda^2\mathbb{R}^{m*}}\big) \circ \frac{1}{2}\{\tau\delta\}\,. \tag{16}$$

Proof. If η_2 is a vector field on W^rP, then $\Theta_r(\eta_2)$ is an $(\mathbb{R}^m \times \mathfrak{w}_m^{r-1}G)$-valued function on W^rP. Thus, if η_1 is another vector field on W^rP, we can consider the derivative $\eta_1\Theta_r(\eta_2)\colon W^rP \to \mathbb{R}^m \times \mathfrak{w}_m^{r-1}G$ of $\Theta_r(\eta_2)$ in the direction of η_1. First we deduce that for every sections σ_1, $\sigma_2\colon M \to LP$ we have

$$\mathcal{W}^r(\bar\sigma_1)\,\Theta_r\big(\mathcal{W}^r(\bar\sigma_2)\big) = \Theta_r\big([\bar{\mathcal{W}}^r\bar\sigma_1, \mathcal{W}^r\bar\sigma_2]\big)\,. \tag{17}$$

Indeed, the rule $\bar\sigma \mapsto \Theta^r(\mathcal{W}^r\bar\sigma)$ is a gauge-natural operator. Hence if commutes with the Lie differentiation, [11]. But the Lie derivative of $\bar\sigma_2$ with respect to $\bar\sigma_1$ is the bracket $[\bar\sigma_1, \bar\sigma_2]$.

Consider now $u \in W^r_x P$ and $Z_1,\ Z_2 \in T_x M,\ \delta(Z_i) = j^r_x \sigma_i,\ i = 1, 2$. Write $u_0 \colon \mathbb{R}^m \to T_x M$ for the underlying map of u. If we interpret $\{D_\Delta \Theta_r\}$ as a map $W^r P \times \mathbb{R}^m \times \mathbb{R}^m \to V^{r-1}_m$, we have

$$\{D_\Delta \Theta_r\}\big(u, u_0^{-1}(Z_1), u_0^{-1}(Z_2)\big) = d\Theta_r\big(\mathcal{W}^r \bar\sigma_1(u), \mathcal{W}^r \bar\sigma_2(u)\big)\,.$$

Applying the classical formula for $d\Theta_r$ and (17), we obtain

$$\begin{aligned} 2d\Theta_r(\mathcal{W}^r\bar\sigma_1, \mathcal{W}^r\bar\sigma_2) &= (\mathcal{W}^r\bar\sigma_1)\Theta_r(\mathcal{W}^r\bar\sigma_2) - (\mathcal{W}^r\bar\sigma_2)\Theta_r(\mathcal{W}^r\bar\sigma_1) \\ &\quad - \Theta_r\big([\mathcal{W}\bar\sigma_1, \mathcal{W}\bar\sigma_2]\big) = \Theta_r\big([\mathcal{W}^r\bar\sigma_1, \mathcal{W}^r\bar\sigma_2]\big)\,. \end{aligned}$$

By the commutativity of (13), the last expression corresponds to $[\![\delta(Z_1), \delta(Z_2)]\!]_r$. □

4. The flow prolongation of principal connections. First we recall a general result on the flow prolongation of connections on an arbitrary fibered manifold $Y \to M$. Let Σ be a general connection on Y considered in the lifting form

$$\Sigma \colon Y \times_M TM \to TY\,.$$

Write $\mathcal{FM}_{m,n}$ for the category of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms. Let F be a bundle functor on $\mathcal{FM}_{m,n}$ of the base order r, [11]. For every vector field X on M we first construct the Σ-lift $\Sigma X \colon Y \to TY$. The flow prolongation $\mathcal{F}(\Sigma X)$ depends on the r-jets of X. This defines a map

$$\mathcal{F}\Sigma \colon FY \times_M J^r TM \to TFY\,.$$

Let Λ be a principal connection on $P^r M$ and $\lambda \colon TM \to J^r TM$ be the corresponding splitting. Then

$$\mathcal{F}(\Sigma, \Lambda) := \mathcal{F}\Sigma \circ (\mathrm{id}_{FY} \times_M \lambda) \colon FY \times_M TM \to TFY$$

is a general connection on FY, that is called the flow prolongation of Σ with respect to Λ, [11].

If we consider a principal connection Γ on a principal bundle $P(M, G)$ in the role of Σ, then $\mathcal{W}^r(\Gamma, \Lambda)$ is a principal connection on $W^r P$. The algebroid form $\gamma \colon TM \to LP$ of Γ is a fibered morphism over id_M. Its r-th jet prolongation is a map $J^r\gamma \colon J^r TM \to J^r LP$. The following assertion will be proved in Section 6 in a more general setting.

Proposition 3. *The algebroid form of* $\mathcal{W}^r(\Gamma, \Lambda)$ *is*

$$J^r\gamma \circ \lambda \colon TM \to J^r LP\,. \tag{18}$$

The first application of (18) is the following assertion, that we deduced in a quite different way in [13]. However, we find remarkable that the algebroid approach reduces the proof to a simple direct evaluation.

Proposition 4. $\mathcal{W}^1(\Gamma,\Lambda)$ *is torsion-free, iff* λ *is torsion-free and* Γ *is curvature-free.*

Proof. By locality, it suffices to discuss the case $P = \mathbb{R}^m \times G$, so that $LP = T\mathbb{R}^m \times \mathfrak{g}$. The sections of LP are pairs (X, ϱ), of a vector field X on $\mathbb{R}^m$ and a map $\varrho\colon \mathbb{R}^m \to \mathfrak{g}$ with the bracket

$$\llbracket (X_1,\varrho_1),(X_2,\varrho_2)\rrbracket = \big([X_1,X_2], X_1\varrho_2 - X_2\varrho_1 + [\varrho_1,\varrho_2]_{\mathfrak{g}}\big)\,, \tag{19}$$

where $[X_1, X_2]$ is the bracket of vector fields and $[\ ,\ ]_{\mathfrak{g}}$ is the bracket in $\mathfrak{g}$.

Consider the canonical coordinates x^i on $\mathbb{R}^m$, the induced coordinates y^i on $T\mathbb{R}^m$ and some linear coordinates z^p on $\mathfrak{g}$. Let y^i_j, z^p_i be the induced coordinates on J^1LP. The map $\llbracket\ ,\ \rrbracket_1$ has the coordinate expression

$$\big(y_1^j y_{2j}^i - y_2^j y_{1j}^i, y_1^i z_{2i}^p - y_2^i z_{1i}^p + c^p_{qr} z_1^q z_2^r\big)\,, \tag{20}$$

where c^p_{qr} are the structure constants of G. Let $\delta\colon TM \to J^1LP$ be a connection of the form

$$z^p = \Delta^p_i(x) y^i\,,\quad y^i_j = \Delta^i_{kj}(x) y^k\,,\quad z^p_i = \Delta^p_{ji}(x) y^j\,. \tag{21}$$

Then the coordinate expression of $\tau\delta$ is

$$\frac{1}{2}\big(\Delta^k_{ij}, \Delta^p_{ij} + c^p_{qr}\Delta^q_i\Delta^r_j\big)\, dx^i \wedge dx^j\,. \tag{22}$$

On the other hand, let γ and λ be of the form

$$z^p = \Gamma^p_i(x) y^i\,,\quad y^i_j = \Lambda^i_{kj}(x) y^k\,. \tag{23}$$

Then the coordinate expression of $J^1\gamma$ is

$$z^p_i = \frac{\partial \Gamma^p_j}{\partial x^i}\, y^j + \Gamma^p_j y^j_i\,. \tag{24}$$

Hence $J^1\gamma \circ \lambda$ is of the form $y^i_j = \Lambda^i_{kj} y^k$ and

$$z^p_i = \Big(\frac{\partial \Gamma^p_j}{\partial x^i} + \Gamma^p_k \Lambda^k_{ij}\Big) y^j\,. \tag{25}$$

By (22), the torsion of $J^1\gamma \circ \lambda$ is

$$\frac{1}{2}\Big(\Lambda^k_{ij}, \frac{\partial \Gamma^p_j}{\partial x^i} + \Gamma^p_k\Lambda^k_{ij} + c^p_{qr}\Gamma^q_i\Gamma^r_j\Big)\, dx^i \wedge dx^j\,. \tag{26}$$

The first term in (26) is the torsion of Λ. If it vanishes, the second term coincides with the algebroid expression

$$\frac{1}{2}\Big(\frac{\partial \Gamma^p_j}{\partial x^i} + c^p_{qr}\Gamma^q_i\Gamma^r_j\Big)\, dx^i \wedge dx^j \tag{27}$$

of the curvature of Γ. □

Now it is easy to prove the general result.

Proposition 5. $\mathcal{W}^r(\Gamma,\Lambda)$ *is torsion-free, iff* Λ *is torsion-free and* Γ *is curvature-free.*

Proof. If $\mathcal{W}^r(\Gamma,\Lambda)$ is torsion-free, then $\mathcal{W}^1(\Gamma,\Lambda)$ is also torsion-free, so that Γ is integrable. Hence there is a local trivialization of P such that $\Gamma^p_i = 0$ identically. Then all non-trivial coefficients of $J^r\gamma$ are also zero and the coordinate expression of $[\![\ ,\ ]\!]_r$ reduces to the case of Λ. So the coordinate expressions of $\tau(J^r\gamma\circ\lambda)$ and $\tau\lambda$ coincide and our assertion follows from Proposition 4.2 in [8]. □

5. Torsion-free connections as reductions. Every $a\in G^1_m$ is a matrix that defines a linear map $l(a)\colon \mathbb{R}^m\to\mathbb{R}^m$. This yields an injection

$$G^1_m \hookrightarrow G^r_m\,,\qquad a\mapsto j^r_0 l(a)\,.$$

In [5], we deduced that the torsion-free connections on P^rM are in bijection with the reductions of $P^{r+1}M$ to the subgroup $G^1_m\subset G^{r+1}_m$. The r-jets $j^r_0\widehat{g}$, $g\in G$, of the constant maps $\widehat{g}\colon\mathbb{R}^m\to G$, $x\mapsto g$, define an injection $G\to T^r_mG$. Then $G^1_m\times G$ is a subgroup of W^r_mG. In [13], we proved that the torsion-free connections on W^1P are in bijection with the reductions of W^2P to $G^1_m\times G$. We are going to deduce such a result for an arbitrary order r.

For every fibered manifold $Y\to M$, the r-th contact morphism is a map $\psi_r\colon TJ^rY\to VJ^{r-1}Y\approx J^{r-1}(VY\to M)$. In the case of a principal bundle $P(M,G)$, we have $VP = P\times\mathfrak{g}$. Then $J^{r-1}VP = J^{r-1}P\times_M J^{r-1}(M,\mathfrak{g})$. Every frame of P^{r-1}_xM identities $J^{r-1}_x(M,\mathfrak{g})$ with the Lie algebra $\mathfrak{t}^{r-1}_mG$ of T^{r-1}_mG If we modify ψ_r in this way, we obtain a map $\bar\psi_r\colon T\mathcal{W}^rP\to\mathfrak{t}^{r-1}_mG$. On the other hand, $\mathfrak{w}^{r-1}_mG = \mathfrak{g}^{r-1}_m\times\mathfrak{t}^{r-1}_mG$, so that we have the product projection $\pi\colon\mathbb{R}^m\times\mathfrak{w}^{r-1}_mG\to\mathfrak{t}^{r-1}_mG$. One verifies directly that

$$\bar\psi_r = \pi\circ\Theta_r\,. \tag{28}$$

We have $J^1(W^rP) = J^1P^rM\times_M J^1J^rP$. In [5], we described an injection $P^{r+1}M\hookrightarrow J^1P^rM$. On the other hand, we have the classical inclusion $J^{r+1}P\hookrightarrow J^1J^rP$. This defines an injection

$$i_r\colon W^{r+1}P\to J^1(W^rP)\,. \tag{29}$$

Let $\Gamma\colon P\to J^1P$ be a connection on $P = W^0P$. The rule

$$\varrho(\Gamma)(u,v) = \big(u,\Gamma(v)\big)\,,\qquad (u,v)\in P^1M\times_M P\,,$$

defines a reduction $\varrho(\Gamma)(P^1M\times_M P)\subset W^1P$ to $G^1_m\times G$, [13]. For a connection $\Delta\colon\mathcal{W}^rP\to J^1\mathcal{W}^rP$, we proceed by the following induction.

Let Δ be such that the underlying connection Δ_1 on $W^{r-1}P$ is torsion-free. Hence Δ_1 defines a reduction

$$\varrho(\Delta_1)\colon P^1M \times_M P \to W^r P$$

by the induction hypothesis.

Proposition 6. *Δ is torsion-free, iff the values of $\Delta \circ \varrho(\Delta_1)$ lie in $i_r(W^{r+1}P)$. Then we define*

$$\varrho(\Delta) = i_r^{-1} \circ \Delta \circ \varrho(\Delta_1)\colon P^1M \times_M P \to W^{r+1}P\,.$$

Proof. Every $\varrho(\Delta)(u,v)$, $(u,v) \in P^1M \times_M P$, represents a linear m-dimensional subspace S in $T\mathcal{W}^r P$, which is identified with a pair of m-dimensional linear subspaces $S_1 \subset TP^rM$ and $S_2 \subset TJ^rP$. By (5) and (28), $d\Theta_r \mid S$ can be considered as the pair $\big(d\varphi_r \mid S_1), d\bar{\psi}_r(S_2)\big)$. By [5], $d\varphi_r \mid S_1 = 0$ if and only if S_1 corresponds to an element of $P^{r+1}M$. Analogously to [6], $d\bar{\psi}_r \mid S_2 = 0$ if and only if S_2 corresponds to an element of $J^{r+1}P$. □

Proposition 7. *Proposition 6 establishes a bijection between the torsion-free connections on W^rP and the reductions of $W^{r+1}P$ to the subgroup $G^1_m \times G \subset W^{r+1}_m G$.*

Proof. On one hand, one verifies directly that $\varrho(\Delta)$ is a reduction to the subgroup $G^1_m \times G$. On the other hand, let $Q\colon P^1M \times_M P \to W^{r+1}P$ be a reduction to the subgroup $G^1_m \times G$. Write $Q_1 = \pi^{r+1}_r \circ Q\colon P^1M \times_M P \to W^rP$. For every $Q_1(u,v) \in W^rP$, $(u,v) \in P^1M \times_M P$, $Q(u,v)$ represents an m-dimensional horizontal subspace of TW^rP. Since our maps are $(G^1_m \times G)$-equivariant, these subspaces are canonically extended into a connection on W^rP. By the proof of Proposition 6, this connection is torsion-free. □

6. The case of W^FP. The r-th jet prolongation of fibered manifolds is a fiber product preserving bundle functor J^r on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered morphisms with local diffeomorphisms as base maps. In [12] we characterized all these functors in terms of Weil algebras, see also [9]. Every such functor F has finite order. In the base order of F is r, then we have an identification $F = (A, H, t)$, where A is a Weil algebra, $H\colon G^r_m \to \operatorname{Aut} A$ is a group homomorphism and $t\colon \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism, provided Aut A means the group of all algebra automorphisms of A and $\mathbb{D}^r_m$ is the Weil algebra $J^r_0(\mathbb{R}^m, \mathbb{R})$. In the case of J^r, we have $A = \mathbb{D}^r_m$, so that $\operatorname{Aut} \mathbb{D}^r_m = G^r_m$, $H = \mathrm{id}_{G^r_m}$ and $t = \mathrm{id}_{\mathbb{D}^r_m}$.

Analogously to the case of J^r, every $F = (A, H, t)$ determines a bundle functor W^F on $\mathcal{PB}_m(G)$

$$W^F P = P^r M \times_M FP\,, \quad W^F f = P^r \underline{f} \times_{\underline{f}} Ff\,,$$

[2]. Similarly to the case of W^r, $W^F(\mathbb{R}^m \times G)$ is a Lie group

$$W^A_H G = G^r_m \rtimes T^A G \tag{30}$$

with the group composition

$$(g_1, C_1)(g_2, C_2) = \big(g_1 \circ g_2, H_G(g_2^{-1})(C_1) \bullet C_2\big)\,, \tag{31}$$

where $\bullet$ denotes the induced group composition in $T^A G$.

Further, $W^F P$ is a principal bundle over M with structure group $W^A_H G$. The values of W^F are in the category $\mathcal{PB}_m(W^A_H G)$.

For every fibered manifold $Y \to M$, t induces a map $t_Y \colon J^r Y \to FY$,

$$t_Y(j^r_x s) = (Fs)(x)\,, \qquad x \in M\,, \tag{32}$$

where s is a local section of Y, which is interpreted as a fibered morphism from the trivial fibered manifold $M \to M$ into Y, so that $Fs \colon M \to FY$. In particular, we have $t_{TM} \colon J^r TM \to FTM$. On the other hand, the anchor map $q \colon LP \to TM$ induces $FLP \to FTM$. In [10] we deduced, by using the theory of semi-direct products, that the Lie algebroid of $W^F P$ is

$$LW^F P = J^r TM \times_{FTM} FLP\,. \tag{33}$$

For every section $\sigma \colon M \to LP$, the vector field $\bar{\sigma}$ on P induces the flow prolongation $\mathcal{W}^F(\bar{\sigma})$, which is a right-invariant vector field on $W^F P$. To found its algebroid form, we use our general idea of the flow natural transformation of F. According to [7], see also [9], for every $Y \to M$ there exists a map

$$\psi^F_Y \colon J^r TM \times_{FTM} F(TY \to M) \to T(FY)$$

with the property that for every projectable vector field η on Y over ξ on M, the flow prolongation $\mathcal{F}\eta$ satisfies

$$\mathcal{F}\eta = \psi^F_Y \circ (j^r \xi \times_{F\xi} F\eta)\,,$$

provided η is considered as a fibered morphism of $TY \to M$ into $TM \to M$. In particular, this yields

Proposition 8. *For every section $\sigma \colon M \to LP$ over $X = q \circ \sigma \colon M \to TM$, the flow prolongation $\mathcal{W}^F(\bar{\sigma})$ corresponds to the section*

$$j^r X \times_{FX} F\sigma \colon M \to LW^F P\,, \quad j^r X \colon M \to J^r TM\,,$$
$$F\sigma \colon M \to F(LP \to M)\,.$$

Let Γ be a connection on P and Λ a connection on P^rM. Hence the flow prolongation $\mathcal{W}^F(\Gamma, \Lambda)$ is a connection on W^FP. The algebroid form $\gamma\colon TM \to LP$ of Γ is a base preserving morphism, so that we can construct $F\gamma\colon FTM \to FLP$. Further, we have $\lambda\colon TM \to J^rTM$. By the very definition of $\mathcal{W}^F(\Gamma, \Lambda)$, we define

Proposition 9. *The algebroid form of $\mathcal{W}^F(\Gamma, \Lambda)$ is $(\lambda, F\gamma \circ t_{TM} \circ \lambda)\colon TM \to LW^FP$.*

In the case $F = J^r$, we have $t_{TM} = \mathrm{id}_{J^rTM}$, so that Proposition 3 is a special case of Proposition 9.

Remark. There is a natural question whether one can define the torsion of connections on an arbitrary principal bundle W^FP. The definition of the canonical form on W^rP is essentially based on the fact that W^{r-1} is the underlying functor of W^r of the order $r-1$. However, Doupovec clarified that the general concept of underlying functors of arbitrary F is rather sophisticated, [1]. So it seems to be reasonable to restrict ourselves to the subfunctors $E \subset J^1F$ with the property that the jet projection $EY \to FY$ is surjective. Then Proposition 2 of [4] implies that there is a canonical form on W^FP with good properties and the procedures of the present paper can be applied.

References

[1] M. Doupovec, *On the underlying lower order bundle functors*, Czechoslovak Math. J. **55** (2005), 901–916.

[2] M. Doupovec, I. Kolář, *Iteration of Fiber Product Preserving Bundle Functors*, Monatsh. Math. **134** (2001), 39–50.

[3] L. Fatibene, M. Francaviglia, *Natural and Gauge Natural Formalism for Classical Fields Theories*, Kluwer, 2003.

[4] I. Kolář, *Generalized G-structures and G-structures of higher order*, Boll. Un. Math. Ital., Suppl. fas. **3** 1975, 249–256.

[5] I. Kolář, *Torsion-free connections on higher order frame bundles*, New Developments in Differential Geometry, Proceedings, Kluwer 1996, 233–241.

[6] I. Kolář, *On holonomicity criteria in second order geometry*, Beiträge zur Algebra und Geometrie **39** (1998), 283–290.

[7] I. Kolář, *On the geometry of fiber product preserving bundle functors*, Diff. Geom. and Its Applications, Proceedings, Silesian University of Opava (2002), 85–92.

[8] I. Kolář, *On the torsion of linear higher order connections*, Central European Journal of Mathematics **3** (2003), 360–366.

[9] I. Kolář, *Weil Bundles as Generalized Jet Spaces*, in: Handbook of Global Analysis, Elsevier, (2007), 625–665.

[10] I. Kolář, A. Cabras, *On the functorial prolongations of principal bundles*, Comment. Math. Univ. Carolin. **47** (2006), 719–731.

[11] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[12] I. Kolář,, W. Mikulski, *On the fiber product preserving bundle functors*, Differential Geometry and Its Applications **11** (1999), 105–115.

[13] I. Kolář,, G. Virsik, *Connections in first principal prolongations*, Suppl. ai Rendiconti del Circolo Matematico di Palermo, Serie II **43** (1996), 163–171.

[14] A. Kumpera, D. Spencer, *Lie equations I*, Princeton University Press, 1972.

[15] K. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press, Cambridge 2005.

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