CONNECTIONS ON HIGHER ORDER FRAME BUNDLES AND THEIR GAUGE ANALOGIES

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To Demeter Krupka, on the occasion of his 65th birthday.

Abstract. In the first part of the paper, we present a survey of the basic properties of connections on the r-th order frame bundle of a manifold. Special attention is paid to the torsion and torsion-free connections. In the second part, connections on the r-th principal prolongation of a principal bundle are treated from similar points of view. The case of the first principal prolongation is discussed in detail.

In the present paper, connection means a principal connection on a principal bundle unless otherwise specified.

Several properties of connections on the r-th order frame bundle P^rM of a manifold M appear in the framework of the general theory of natural bundles and operators. This is described in the book by D. Krupka and J. Janyška, [21], and in the monograph [18]. So the first part of the present paper is devoted to a survey of some more specific properties of connections on P^rM that are mostly related with the idea of torsion. In Section 1 we underline that the Lie algebroid version of a connection on the principal bundle $P^rM(M, G^r_m)$ is a linear r-th order connection on TM. So we have two different approaches to the concept of torsion. Proposition 2 reads that both approaches are equivalent.

In Section 3 we clarify that the torsion-free connections on P^rM are in bijection with the reductions of $P^{r+1}M$ to the canonical injection of G^1_m into G^{r+1}_m. This enables us to define the r-th exponential operator transforming every torsion-free connection Λ on P^1M into a torsion-free connection on P^rM. In particular, this implies that Λ determines a general connection on every natural bundle over m-manifolds. In Section 5, the Lie algebroid construction of the exponential operator

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is based on two interesting lemmas concerning the r-jet of the commutator of vector fields on M. Then we deduce that the torsion-free connections on P^rM are in bijection with the splittings from the cotangent bundle T^*M into the bundle of $(1, r+1)$-covelocities on M.

In Section 7 we discuss a connection on P^rM from the viewpoint of the theory of higher order G-structures and we characterize its integrability in this sense. In Section 8 we present a recent result by W. Mikulski, [27], who determined all natural operators transforming a torsion-free connection on P^1M into a connection on P^rM. Section 9 is devoted to the basic properties of semiholonomic 2-jets, that represent a useful tool for several problems of the present paper.

The principal prolongation W^rP of an arbitrary principal bundle $P(M, G)$ is defined in Section 10 in a formally slightly different way to [18]. We hope this could be useful in applications. Then we summarize the basic properties of connections on W^rP and their torsions. In Section 11 we clarify that every connection Φ on W^1P is canonically identified with the triple (Γ, Λ, D) of a connection Γ on P, a connection Λ on P^1M and a section D of $L^0P \otimes \overset{2}{\bigotimes} T^*M$, where L^0P is the adjoint bundle of P. In Section 12 we present the list of all gauge-natural operators transforming every pair (Γ, Λ) of a connection Γ on P and a torsion-free connection Λ on P^1M into a connection on W^1P. Finally we outline how the semiholonomic 2-jets can be used in the theory of connections on W^1P. In particular, we introduce the conjugate connection $\tilde{\Phi}$ to every connection Φ on W^1P by using the canonical involution of semiholonomic 2-jets.

All manifolds and maps are assumed to be infinitely differentable. Unless otherwise specified, we use the terminology and notations from the book [18].

1. The algebroid form of connections on P^rM. First we recall that the Lie algebroid $LP \to M$ of an arbitrary principal bundle $P(M, G)$ is defined by $LP = TP/G$. So the elements of LP are the right invariant families of tangent vectors along the individual fibers of P, every section $\sigma\colon M \to LP$ is identified with a right invariant vector field $\overline{\sigma}\colon P \to TP$ and the bracket $[\overline{\sigma}_1, \overline{\sigma}_2]$ of right invariant vector fields on P induces the bracket $[\![\sigma_1, \sigma_2]\!]$ of LP. The canonical projection $q\colon LP \to TM$ is called the anchor map. Clearly, a connection Γ on P can be interpreted as a linear morphism $\gamma\colon TM \to LP$ satisfying $q \circ \gamma = \mathrm{id}_{TM}$, [26].

We write P^rM for the r-th order frame bundle of an m-dimensional manifold M. This is a principal bundle over M with structure group $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$. Every local diffeomorphism $f\colon M_1 \to M_2$

induces a principal bundle morphism $P^r f\colon P^r M_1 \to P^r M_2$, so that P^r is a bundle functor on the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms, [18]. For every vector field $X\colon M \to TM$, its flow prolongation

$$\mathcal{P}^r X := \frac{\partial}{\partial t}\Big|_0 P^r(Fl_t^X) \tag{1}$$

is a right invariant vector field on $P^r M$. This follows directly from the fact that the values of P^r are in the category $\mathcal{PB}_m(G_m^r)$ of principal G_m^r-bundles over m-manifolds and their local principal bundle isomorphisms. Since P^r is an r-th order bundle functor, the restriction $\mathcal{P}^r X \mid P_x^r M$ depends on $j_x^r X$ only, $x \in M$.

Proposition 1. *The rule*

$$I_M^r(j_x^r X) = \mathcal{P}^r X \mid P_x^r M \tag{2}$$

identifies $J^r TM$ with $LP^r M$.

Proof. We have to prove that I_M^r is a diffeomorphism. But $P^r M = \operatorname{reg} T_m^r M$ is an open subset of the bundle $T_m^r M$ of all (m,r)-velocities on M and $\mathcal{P}^r X$ is the restriction of the flow prolongation $\mathcal{T}_m^r X$ to this subset. Hence the bijectivity of I_M^r follows from the existence of an exchange isomorphism $\varkappa_M\colon T_m^r TM \to TT_m^r M$ such that $\mathcal{T}_m^r X = \varkappa_M \circ T_m^r M$, [16], [18]. □

A linear r-th order connection on TM is a linear morphism $TM \to J^r TM$ that splits the target jet projection. According to Proposition 1, every connection $\Gamma\colon P^r M \to J^1 P^r M$ is identified with a linear splitting $\gamma\colon TM \to J^r TM$. We say that γ is the algebroid form of Γ.

2. Two approaches to the torsion on $P^r M$. The canonical $(\mathbb{R}^m \times \mathfrak{g}_m^{r-1})$-valued 1-form φ_r on $P^r M$ is defined as follows. We have $\mathbb{R}^m \times \mathfrak{g}_m^{r-1} = T_{e_{r-1}} P^{r-1}\mathbb{R}^m$, where $e_{r-1} = j_0^{r-1} \operatorname{id}_{\mathbb{R}^m}$. Every $u = j_0^r f$, $f\colon \mathbb{R}^m \to M$, induces $P^{r-1} f\colon P^{r-1}\mathbb{R}^m \to P^{r-1} M$. The tangent map $\tilde{u} := T_{e_{r-1}} P^{r-1} f\colon T_{e_{r-1}} P^{r-1}\mathbb{R}^m \to T_{u_{r-1}} P^{r-1} M$, $u_{r-1} = \pi_{r-1}^r(u)$, depends on u only. Then one defines

$$\varphi_r(A) = \tilde{u}^{-1}\big(T\pi_{r-1}^r(A)\big)\,, \qquad A \in T_u P^r M\,.$$

P. C. Yuen introduced the torsion of a connection Γ on $P^r M$ as the exterior covariant differential $D_\Gamma \varphi_r$ of φ_r, [33]. Since $D_\Gamma \varphi_r$ is a horizontal 2-form on $P^r M$, it can be interpreted as a map $P^r M \to (\mathbb{R}^m \times \mathfrak{g}_m^{r-1}) \otimes \Lambda^2 T^* M$. Taking into account the identification $\tilde{u}_1\colon \mathbb{R}^m \to T_x M$, $u_1 = \pi_1^r(u)$, we construct

$$\overline{D}_\Gamma \varphi_r\colon P^r M \to (\mathbb{R}^m \times \mathfrak{g}_m^{r-1}) \otimes \Lambda^2 \mathbb{R}^{m*}\,. \tag{3}$$

On the other hand, the $(r-1)$-jet at $x \in M$ of the bracket $[\xi, \eta]$ of two vector fields ξ, η on M depends on the r-jets $j^r_x\xi$ and $j^r_x\eta$. This defines a map

$$[\,,\,]_{r-1}\colon J^rTM \times_M J^rTM \to J^{r-1}TM\,.$$

Let $\gamma\colon TM \to J^rTM$ be the algebroid form of Γ. According to A. Zajtz, [28], the torsion of γ is a map $\tau\gamma\colon TM \times_M TM \to J^{r-1}TM$ defined by

$$\tau\gamma(A,B) = \big[\gamma(A), \gamma(B)\big]_{r-1}\,, \qquad A, B \in T_xM\,. \tag{4}$$

Clearly, $\tau\gamma$ can be interpreted as a section of $J^{r-1}TM \otimes \Lambda^2T^*M$. This is a fiber bundle associated to P^rM with standard fiber $(\mathbb{R}^m \times \mathfrak{g}^{r-1}_m) \otimes \Lambda^2\mathbb{R}^{m*}$. So the frame form of $\tau\gamma$ is a map

$$\overline{\tau\gamma}\colon P^rM \to (\mathbb{R}^m \times \mathfrak{g}^{r-1}_m) \otimes \Lambda^2\mathbb{R}^{m*}\,.$$

In [14], we deduced

Proposition 2. *We have* $\overline{D}_\Gamma\varphi_r = \frac{1}{2}\overline{\tau\gamma}$.

Let Γ and Δ be two connections on P^rM over the same connection on $P^{r-1}M$. Consider their algebroid forms $\gamma, \delta\colon TM \to J^rTM$. Since the kernel of $\pi^r_{r-1}\colon J^rTM \to J^{r-1}TM$ is $TM \otimes S^rT^*M$, [18], the difference of γ and δ is a section

$$\gamma - \delta\colon M \to TM \otimes S^rT^*M \otimes T^*M\,. \tag{5}$$

Proposition 3. *If both γ and δ are torsion-free, then the values of $\gamma - \delta$ lie in $TM \otimes S^{r+1}T^*M$.*

Proof. If x^i are some local coordinates on M, $X^i = dx^i$ are the induced coordinates on TM and X^i_α are the jet coordinates on J^rTM, then the equations of γ or δ are

$$X^i_\alpha = \Gamma^i_{\alpha j}(x)X^j \quad \text{or} \quad X^i_\alpha = \Delta^i_{\alpha j}(x)X^j\,, \qquad 1 \le |\alpha| \le r\,,$$

where α is a multi-index of range m. The difference $\gamma - \delta$ can be interpreted as a map $TM \times_M TM \to TM \otimes S^{r-1}T^*M$ of the form

$$\big(\Gamma^i_{\beta jk} - \Delta^i_{\beta jk}\big)\xi^j\eta^k\,, \qquad |\beta| = r-1\,.$$

The only r-th order terms in $j^{r-1}[\xi, \eta]$ are

$$\xi^j\frac{\partial^r\eta^i}{\partial x^j\partial_\beta x} - \eta^j\frac{\partial^r\xi^i}{\partial x^j\partial_\beta x}\,, \qquad |\beta| = r-1\,. \tag{6}$$

If γ is torsion-free, then (6) yields $\Gamma^i_{\beta jk} = \Gamma^i_{\beta kj}$. If δ is also torsion-free, (5) is symmetric in the last two subscripts. □

From the proof one sees directly that $\gamma - \delta$ is an arbitrary section of $TM \otimes S^{r+1}T^*M$.

3. Torsion-free connections on P^rM as reductions of $P^{r+1}M$. Every $a \in G^1_m$ is a matrix, which defines a linear map $l(a)\colon \mathbb{R}^m \to \mathbb{R}^m$. This induces a group homomorphism

$$l_{r-1}\colon G^1_m \to G^r_m\,, \qquad l_{r-1}(a) = j^r_0 l(a)\,.$$

S. Kobayashi proved, [8], that the torsion-free connections on P^1M are in bijection with the reductions of P^2M to the subgroup $l_1(G^1_m) \subset G^2_m$. We deduce an analogous result for arbitrary order r. This is based on the following injection $i^r_M\colon P^{r+1}M \hookrightarrow J^1P^rM$. Every $u = j^{r+1}_0 f \in P^{r+1}M$ determines a local section ψ of $P^rM \to M$

$$\psi(y) = j^r_0\big(f \circ t_{f^{-1}(y)}\big)\,, \tag{7}$$

where y lies in a neighbourhood of $f(0) \in M$ and $t_{f^{-1}(y)}\colon \mathbb{R}^m \to \mathbb{R}^m$ is the translation $x \mapsto x + f^{-1}(y)$. Then we set $i^r_M(u) = j^1_{f(0)}\psi$. If x^i, $x^i_j, \dots, x^i_{j_i\dots j_r}$ are the standard coordinates on $P^r\mathbb{R}^m$, $x^i_{j,k}, \dots, x^i_{j_1\dots j_r,k}$ are the induced coordinates on $J^1P^r\mathbb{R}^m$ and $x^i_{j_1\dots j_{r+1}}$ are the additional coordinates on $P^{r+1}\mathbb{R}^m$, then (7) implies directly the following coordinate form of i^r

$$x^i_{j,k} = x^i_{jl}\tilde{x}^l_k, \dots, x^i_{j_1\dots j_r,k} = x^i_{j_1\dots j_r l}\tilde{x}^l_k\,, \tag{8}$$

where $\tilde{x}^i_j$ is the inverse matrix to x^i_j.

Every $X \in J^1P^rM$ over $\beta X \in P^rM$ is identified with an m-plane in the tangent space $T_{\beta X}P^rM$, which will be denoted by the same symbol X. Hence we can consider the restriction $d\varphi_r \mid X$ of the exterior differential of φ_r to X. Denote by $X_1 \in J^1P^{r-1}M$ the underlying element of X. The following lemma from [14] is close to a result by Yuen, [33].

Lemma 1. *Let $X \in J^1P^rM$ satisfy $X_1 = i^{r-1}_M(\beta X)$. Then $X \in i^r_M(P^{r+1}M)$ if and only if $d\varphi_r \mid X = 0$.*

For $r = 1$ we have no X_1 and the claim $d\varphi_1 \mid X = 0$ if and only if $X \in i^1_M(P^2M)$ was used in [8].

For every torsion-free connection Γ on P^rM we define a map $\mu(\Gamma)\colon P^1M \to P^{r+1}M$ by the following induction. Consider a connection $\Gamma\colon P^rM \to J^1P^rM$ such that the underlying connection $\Gamma_1\colon P^{r-1}M \to J^1P^{r-1}M$ is torsion-free, so that Γ_1 determines a map $\mu(\Gamma_1)\colon P^1M \to P^rM$ be the induction hypothesis.

Proposition 4. Γ *is torsion-free, if and only if the values of* $\Gamma \circ \mu(\Gamma_1)$ *lie in* $i^r_M(P^{r+1}M)$.

Then we define $\mu(\Gamma) = (i^r_M)^{-1} \circ \Gamma \circ \mu(\Gamma_1)\colon P^1M \to P^{r+1}M$.

Proof. By Lemma 1, we have $d\varphi_r \mid X = 0$ for all $X \in \Gamma\big(\mu(\Gamma_1)(P^1M)\big)$. But φ_r is a pseudotensorial form, [18], so that $d\varphi_r \mid A = 0$ holds for every $A \in \Gamma(P^rM)$. This is equivalent to $D_\Gamma\varphi_r = 0$. □

For every principal bundle $P(M, G)$, we have an induced right action of G on J^1P, $\big(j^1_x s(y), g\big) \mapsto j^1_x\big(s(y)g\big)$, where s is a local section of P on a neighbourhood of $x \in M$ and $g \in G$. This action will be denoted by $(X, g) \mapsto X\varrho(g)$.

Lemma 2. *For every $v \in P^{r+1}M$ and $a \in G^1_m$, we have*

$$i^r_M\big(vl_r(a)\big) = i^r_M(v)\varrho\big(l_{r-1}(a)\big)\,.$$

Proof. If $v = j^{r+1}_0 f$, then $i^r_M(v)\varrho\big(l_{r-1}(a)\big) = j^1_x\big[j^r_0\big(f \circ t_{f^{-1}(y)} \circ l(a)\big)\big]$. On the other hand, $i^r_M\big(vl_r(a)\big) = j^1_x\big[j^r_0\big(f \circ l(a) \circ t_{l(a)^{-1}(f^{-1}(y))}\big)\big]$. But $t_z \circ l(a) = l(a) \circ t_{l(a)^{-1}(z)}$, $z \in \mathbb{R}^m$, is a well known relation from the affine geometry. □

By Lemma 2, $\mu(\Gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to the subgroup $l_r(G^1_m) \subset G^{r+1}_m$. Indeed, using induction we obtain

$$\mu(\Gamma)(ua) = (i^r_M)^{-1}\big[\Gamma\big(\mu(\Gamma_1)(u)\big)\varrho\big(l_{r-1}(a)\big)\big] = \mu(\Gamma)(u)l_r(a)\,.$$

On the other hand, every reduction $Q \subset P^{r+1}M$ to the subgroup $l_r(G^1_m)$ induces a map (denoted by the same symbol) $Q\colon P^1M \to P^{r+1}M$ as follows. For every $v \in Q$ we construct $u = \pi^{r+1}_1(v)$ and we set $Q(u) = v$. Any other $\bar v$ in the same fiber of $Q \to M$ is of the form $\bar v = vl_r(a)$, $a \in G^1_m$. This implies $\pi^{r+1}_1(\bar v) = ua$, so that our definition is correct.

Proposition 5. *Proposition 4 establishes a bijection between torsion-free connections on P^rM and reductions of $P^{r+1}M$ to $l_r(G^1_m)$.*

Proof. First we deduce that $\mu(\Gamma)\colon P^1M \to P^{r+1}M$ is a reduction to $l_r(G^1_m)$. For every $u \in P^1M$ and $a \in G^1_m$ we have

$$\begin{aligned}\mu(\Gamma)(ua) &= (i^r_M)^{-1}\big[\Gamma\big(\mu(\Gamma_1)(ua)\big)\big] = (i^r_M)^{-1}\big[\Gamma\big(\mu(\Gamma_1)(u)l_{r-1}(a)\big)\big]\\ &= (i^r_M)^{-1}\big[\Gamma\big(\mu(\Gamma_1)(u)\big)\varrho\big(l_{r-1}(a)\big)\big] = (i^r_M)^{-1}\big[(i^r_M)\big(\mu(\Gamma)(u)\big)l_r(a)\big)\big]\end{aligned}$$

by definition, by the induction hypothesis, by right-invariance of Γ and by Lemma 2. Conversely, if $Q\colon P^1M \to P^{r+1}M$ is a reduction to $l_r(G^1_m)$, then $Q_1 = \pi^{r+1}_r \circ Q\colon P^1M \to P^rM$ is a reduction to $l_{r-1}(G^1_m)$. We define $\Gamma\colon Q_1(P^1M) \to J^1P^rM$ by $\Gamma\big(Q_1(u)\big) = i^r_M\big(Q(u)\big)$. By Lemma 2, it holds $\Gamma\big(Q_1(ua)\big) = i^r_M\big(Q(ua)\big) = i^r_M\big(Q(u)l_r(a)\big) = i^r_M\big(Q(u)\big)\varrho\big(l_{r-1}(a)\big) = \Gamma\big(Q_1(u)\big)\varrho\big(l_{r-1}(a)\big)$. Hence Γ is a right-invariant map, which is canonically extended into a connection on P^rM. □

4. The r-th order exponential prolongation. The following construction represents an interesting application of Proposition 5. Consider a torsion-free connection Λ on P^1M. For every $x \in M$, Λ determines the exponential map $\exp_x^\Lambda \colon U_x \to M$, where $U_x \subset T_xM$ is a neighbourhood of the origin. Then we define a map $E_r(\Lambda)\colon P^1M \to P^{r+1}M$ by

$$E_r(\Lambda)(u) = j_0^{r+1}(\exp_x^\Lambda \circ u)\,, \qquad u \in P^1_xM\,, \tag{9}$$

where u is interpreted as a map $\mathbb{R}^m \to T_xM$.

Proposition 6. *$E_r(\Lambda)(P^rM)$ is a reduction of $P^{r+1}M$ to $l_r(G^1_m)$.*

Proof. For all $u \in P^1M$ and $a \in G^1_m$, we have $E_r(\Lambda)(ua) = j_0^{r+1}\big(\exp_x^\Lambda \circ u \circ l(a)\big) = E_r(\Lambda)(u)l_r(a)$. □

By Proposition 5, $E_r(\Lambda)$ is a torsion-free connection on P^rM, that is called the r-th exponential prolongation of Λ. The rule $\Lambda \mapsto E_r(\Lambda)$ is said to be the r-th order exponential operator on the bundle $Q_\tau P^1M$ of torsion-free connections on P^1M.

W. Mikulski invented another construction of the exponential prolongation, [27]. Every $X \in T_xM$ is extended into a vector field $\tilde{X}$ on T_xM by means of translations. The exponential map $\exp_x^\Lambda$ transforms $\tilde{X}$ locally into a vector field $(\exp_x^\Lambda)_*(\tilde{X})$ on M. Then we can construct

$$\varepsilon_r(\Lambda)(X) = j_0^r\big((\exp_x^\Lambda)_*(\tilde{X})\big) \in J^r_xTM\,. \tag{10}$$

In Section 5 we deduce that $\varepsilon_r(\Lambda)$ is the algebroid form of $E_r(\Lambda)$.

This result enables us to describe another geometrically interesting construction of $E_r(\Lambda)$. The flow prolongation $\mathcal{P}^r\big((\exp_x^\Lambda)_*(\tilde{X})\big)$ is a vector field on P^rM. By Section 1, the lifting map $P^rM \times_M TM \to TP^rM$ of $E_r(\Lambda)$ is

$$E_r(\Lambda)(u, X) = \mathcal{P}^r\big((\exp_x^\Lambda)_*(\tilde{X})\big)(u)\,, \qquad u \in P^r_xM\,.$$

Further, consider an r-th order natural bundle F over m-manifolds, [18]. So FM is a fiber bundle associated to P^rM with standard fiber $F_0\mathbb{R}^m$. Every principal connection Γ on P^rM induces a general connection Γ_F on P^rM. We shall use the construction of Γ_F by means of lifting vector fields. In general, every right-invariant vector field Z on P^rM induces a vector field Z_F on FM as follows. If $Z(u) = \frac{dc(0)}{dt}$, $c\colon \mathbb{R} \to P^rM$, $u = c(0)$, then

$$Z_F\big(\{u, a\}\big) = \frac{d}{dt}\Big|_0 \big\{c(t), a\big\}\,, \qquad a \in F_0\mathbb{R}^m\,.$$

Since Z is right-invariant, this definition is correct. Then the Γ_F-lift of a vector field X on M is prescribed by $\Gamma_F(X) = (\Gamma X)_F$.

Clearly, the flow prolongation $\mathcal{F}X$ of a vector field X on M with respect to F satisfies $\mathcal{F}X = (\mathcal{P}^r X)_F$. Thus we have deduced

Proposition 7. *For every r-th order natural bundle F, the rule*

$$E_r(\Lambda)_F(v, X) = \mathcal{F}\big((\exp_x^\Lambda)_*(\tilde{X})\big)(v)\,, \qquad v \in F_x M,\ X \in T_x M$$

transforms every torsion-free connection Λ on P^1M into general connection $E_r(\Lambda)_F$ on FM.

5. The exponential prolongation in the algebroid form. Consider an arbitrary linear splitting $\gamma\colon TM \to J^rTM$. For a linear frame $u \in P^1_xM$, $u = (A_1, \dots, A_m)$, $A_i \in T_xM$, we take vector fields X_i satisfying $j^r_x X_i = \gamma(A_i)$, $i = 1, \dots, m$. Then

$$\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(x)$$

is a local map $\mathbb{R}^m \to M$ and we define

$$\sigma(\gamma)(u) = j^{r+1}_0\big(Fl^{X_1}_{t^1} \circ \dots \circ Fl^{X_m}_{t^m}\big)(x) \in P^{r+1}_x M\,. \tag{11}$$

One verifies easily that $\sigma(\gamma)(u)$ depends on u and γ only.

Proposition 8. *If γ is torsion-free, then $\sigma(\gamma)(P^1M)$ is a reduction of $P^{r+1}M$ to $l_r(G^1_m)$.*

Proof is based, in a very instructive way, on the definition (4) of $\tau\gamma$. We shall use the following two lemmas from [15].

Consider two vector fields X and Y on M. Then $\big(Fl^X_t \circ Fl^Y_\tau\big)(x)$ is a local map $\mathbb{R}^2 \to M$, so that $j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) \in (T^{r+1}_2 M)$ is a $(2, r+1)$-velocity on M.

Lemma 3. *If $j^{r-1}_x[X, Y] = 0$, then*

$$j^{r+1}_{0,0}\big(Fl^X_t \circ Fl^Y_\tau\big)(x) = j^{r+1}_{0,0}\big(Fl^Y_\tau \circ Fl^X_t\big)(x)\,. \tag{12}$$

Further, $\big(Fl^X_t \circ Fl^Y_t\big)(x)$ is a local map $\mathbb{R} \to M$, so that $j^{r+1}_0\big(Fl^X_t \circ Fl^Y_t\big)(x) \in (T^{r+1}_1 M)_x$ is a $(1, r+1)$-velocity on M.

Lemma 4. *If $j^{r-1}_x[X, Y] = 0$, then*

$$j^{r+1}_0\big(Fl^X_t \circ Fl^Y_t\big)(x) = j^{r+1}_0\big(Fl^{X+Y}_t\big)(x)\,. \tag{13}$$

We shall also apply the well known formula

$$Fl^X_{ct} = Fl^{cX}_t\,, \qquad c \in \mathbb{R}\,. \tag{14}$$

Take $a = (a^i_j) \in G^1_m$ and consider $ua = (a^j_i A_j)$. Since γ is torsion-free, by (13), (14) and (12) we obtain gradually

$$\begin{aligned}\sigma(\gamma)(ua) &= j_0^{r+1}\big(Fl_{t^1}^{a_1^1X_1+\dots+a_1^mX_m}\circ\dots\circ Fl_{t^m}^{a_m^1X_1+\dots+a_m^mX_m}\big)\\ &= j_0^{r+1}\big(Fl_{t^1}^{a_1^1X_1}\circ\dots\circ Fl_{t^1}^{a_1^mX_m}\circ\dots\circ Fl_{t^m}^{a_m^1X_1}\circ\dots\circ Fl_{t^m}^{a_m^mX_m}\big)\\ &= j_0^{r+1}\big(Fl_{a_1^1t^1}^{X_1}\circ\dots\circ Fl_{a_1^mt^1}^{X_m}\circ\dots\circ Fl_{a_m^1t^m}^{X_1}\circ\dots\circ Fl_{a_m^mt^m}^{X_m}\big)\\ &= j_0^{r+1}\big(Fl_{a_1^1t^1+\dots+a_m^1t^m}^{X_1}\circ\dots\circ Fl_{a_1^mt^1+\dots+a_m^mt^m}^{X_m}\big)\,.\end{aligned}$$

This proves Proposition 8.

To clarify the relation of $\sigma(\gamma)$ to the reduction $\mu(\Gamma)$ from Section 3, we need the following form of the injection $i^r_M\colon P^{r+1}M \to J^1P^rM$. We have $P^rM \subset T^r_mM$. Clearly, $j^r_0f \in T^r_mM$, $f\colon \mathbb{R}^m \to M$, can be expressed in the form

(15) $$j^r_0f = (T^r_mf)(e_r)\,, \qquad e_r = j^r_0\,\mathrm{id}_{\mathbb{R}^m}\ .$$

Write $E_i = \frac{\partial}{\partial t}\big|_0 j^r_0\tau^i_t \in T_{e_r}T^r_m\mathbb{R}^m$, where $\tau^i_t\colon \mathbb{R}^m \to \mathbb{R}^m$ is the translation $\bar t^1 = t^1, \dots, \bar t^i = t^i + t, \dots, \bar t^m = t^m$. If we consider $j^{r+1}_0\psi \in P^{r+1}M$, then

(16) $$(TT^r_m\psi)(E_i)$$

is an m-tuple of tangent vectors at $j^r_0\psi \in P^rM$. The linear span of these vectors defines $i^r_M\big(j^{r+1}_0\psi\big) \in J^1P^rM$.

Proposition 9. *If* γ *is torsion-free and* Γ *is the corresponding connection on* P^rM*, then* $\sigma(\gamma) = \mu(\Gamma)$.

Proof. We proceed by introduction. If γ_1 and Γ_1 are the underlying connections in the order $r-1$, then $\sigma(\gamma_1) = \mu(\Gamma_1)$ by the induction hypothesis. Consider $u = (A_1, \dots, A_m) \in P^1_xM$ and write

$$v = \sigma(\gamma_1)(u) = \mu(\Gamma_1)(u)\,.$$

By (16), $i^r_M\big(j^{r+1}_0\big(Fl^{X_1}_{t^1}\circ\dots\circ Fl^{X_m}_{t^m}\big)(x)\big)$ is the linear span of the vectors

(17) $$TT^r_m\big(Fl^{X_1}_{t^1}\circ\dots\circ Fl^{X_m}_{t^m}\big)(E_i)\,, \qquad i = 1, \dots, m\,.$$

Using the basic properties of flows, Lemma 3 and (15), we deduce that (17) is equal to

$$\begin{aligned}&\frac{\partial}{\partial t}\Big|_0 T^r_m\big(Fl^{X_1}_{t^1}\circ\dots\circ Fl^{X_i}_{t+t^i}\circ\dots\circ Fl^{X_m}_{t^m}\big)(e_r)\\ &\qquad = \frac{\partial}{\partial t}\Big|_0\big(Fl^{\mathcal{T}^r_mX_i}_{t}\circ Fl^{\mathcal{T}^r_mX_1}_{t^1}\circ\dots\circ Fl^{\mathcal{T}^r_mX_m}_{t^m}\big)(e_r)\\ &\qquad = \mathcal{T}^r_mX_i\big(T^r_m(Fl^{X_1}_{t^1}\circ\dots\circ Fl^{X_m}_{t_m})(e_r)\big) = \mathcal{T}^r_mX_i(v)\,.\end{aligned}$$

By (2) and by the induction hypothesis, this m-tuple spans $\mu(\Gamma)(v)$. $\square$

From the proof of Proposition 8 we obtain easily that the construction (10) of $\varepsilon_r(\Lambda)$ by W. Mikulski is the algebroid form of the exponential prolongation $E_r(\Lambda)$ introduced in Section 4.

6. Splittings $T^*M \to T^{r+1*}M$. The space $T^{r+1*}M = J^{r+1}(M,\mathbb{R})_0$ of all $(1, r+1)$-covelocities on M is a vector bundle, [18]. By a splitting $s\colon T^*M \to T^{r+1*}M$ we mean a linear morphism satisfying $\pi_1^{r+1} \circ s = \mathrm{id}_{T^*M}$. We remark that such splittings play an interesting role in the construction of Poincaré-Cartan morphisms in the higher order variational calculus, [11].

Proposition 10. *There is a canonical bijection between reductions $Q \subset P^{r+1}M$ to subgroup $l_r(G_m^1)$ and splittings $s\colon T^*M \to T^{r+1*}M$.*

Proof. Every $b \in T_x^*M$ determines a linear map $\lambda(b)\colon T_xM \to \mathbb{R}$. Let $v = j_0^{r+1}f \in Q_x$, so that $u = \pi_1^{r+1}(v) \in P_x^1M$ can be interpreted as a map $u\colon \mathbb{R}^m \to T_xM$. Then we set

$$s(b) = j_x^{r+1}\big[\lambda(b) \circ u \circ f^{-1}\big] \in T_x^{r+1*}M\,. \tag{18}$$

Since Q is a reduction to $l_r(G_m^1)$, (18) does not depend on the choice of $v \in Q_x$, The fact that s is a splitting follows directly from (18). Conversely, let $s\colon T^*M \to T^{r+1*}M$ be a splitting. A frame $u \in P_x^1M$ is a basis $(e_1, \dots, e_m)$ of T_xM. Consider the dual basis $u^* = (e^1, \dots, e^m)$ of T_x^*M. Then $s(e^1), \dots s(e^m)$ are the components of an $(r+1)$-jet $s(u^*) \in J_x^{r+1}(M,\mathbb{R})_0$. Write $Q(u) = \big(s(u^*)\big)^{-1} \in P_x^{r+1}M$ for the inverse jet. If we take $ua = (a_i^j e_j)$, then $(ua)^* = (\tilde a_j^i e^j)$, where $\tilde a_j^i$ is the inverse matrix to a_j^i. Hence $s\big((ua)^*\big) = \tilde a_j^i s(e^j) = l_r(a^{-1}) \circ s(u^*)$, which implies $Q(ua) = Q(u)l_r(a)$. Finally, one verifies easily that the maps $Q \mapsto s$ and $s \mapsto Q$ are inverse each other. □

Taking into account Proposition 5, we obtain a canonical bijection between torsion-free connections on P^rM and splittings $T^*M \to T^{r+1*}M$.

We remark that Proposition 10 yields another proof of Proposition 3.

7. The viewpoint of higher order G-structures. A connection on P^rM can be viewed as a kind of higher order G-structure on M. We recall that a k-th order G-structure on M is said to be integrable, if it is locally isomorphic to the product $\mathbb{R}^m \times G$, where $G \subset G_m^k$ is the structure group. We are going to apply this approach to the algebroid form $\gamma\colon TM \to J^rTM$. (We remark that this kind of integrability plays an important role in our theory of the flow prolongation of some tangent valued forms, [1].)

On $\mathbb{R}^m$, there is a distinguished connection $C_r\colon T\mathbb{R}^m \to J^rT\mathbb{R}^m$ defined by

$$C_r(X) = j^r_x\tilde{X}\,, \qquad X \in T_x\mathbb{R}^m\,,$$

where $\tilde{X}$ is the constant vector field on $\mathbb{R}^m$ constructed from X by means of translations.

Definition 1. We say that $\gamma\colon TM \to J^rTM$ is integrable, if for every $x \in M$ there exists a neighbourhood U and a diffeomorphism $f\colon U \to \mathbb{R}^m$ satisfying $C_r \circ Tf = J^rTf \circ (\gamma \mid U)$.

Clearly, every integrable connection γ is torsion-free. According to a classical result, a connection Λ on P^1M is integrable, iff it is both torsion-free and curvature-free. Then every exponential prolongation $\varepsilon_k(\Lambda)$ is also integrable.

Thus, the torsion of γ is the first obstruction to its integrability. Consider the underlying connections $\gamma_k = \pi^r_k \circ \gamma$, $k = 1,\dots,r$. Clearly, if γ is torsion-free or integrable, then each γ_k is so. Assume that γ is torsion-free. Then the curvature of γ_1 is the second obstruction to the integrability of γ. If this curvature vanishes, each connection $\varepsilon_k(\gamma_1)$ is integrable. The difference

$$\gamma_2 - \varepsilon_2(\gamma_1)$$

is a tensor field of type $TM \otimes S^3T^*M$ that is the third obstruction to the integrability of γ. Assume by induction that the first up to $(k+1)$-st obstruction to the integrability of γ vanish. Then $\gamma_k = \varepsilon_k(\gamma_1)$ and the tensor field of type $TM \otimes S^{k+2}T^*M$

$$\gamma_{k+1} - \varepsilon_{k+1}(\gamma_1)$$

is the next obstruction to the integrability of γ. If all these $r+1$ obstructions vanish, then $\gamma = \varepsilon_r(\gamma_1)$ is integrable. Thus, we have proved

Proposition 11. *γ is integrable if and only if all the following conditions are satisfied*

a) *γ is torsion-free,*

b) *γ_1 is curvature-free,*

c) *all the gradually defined tensor fields $\gamma_k - \varepsilon_k(\gamma_1)$, $k = 2,\dots,r$, vanish.*

8. Natural operators $C^\infty Q_\tau P^1M \to C^\infty QP^rM$. We write QP for the connection bundle of an arbitrary principal bundle $P(M,G)$, [18]. The connections on P form the space $C^\infty QP$ of all sections of $QP \to M$. Further, we write $Q_\tau P^rM$ for the bundle of all torsion-free connections on P^rM, [14]. So the r-th exponential operator on M is a natural operator $C^\infty Q_\tau P^1M \to C^\infty Q_\tau P^rM$. Using E_r, W. Mikulski solved a rather sophisticated problem of finding all natural operators $C^\infty Q_\tau P^1M \to C^\infty QP^rM$ and $C^\infty Q_\tau P^1M \to C^\infty Q_\tau P^rM$, [27].

Every torsion-free connection Λ on P^1M defines a vector bundle isomorphism

$$\psi_\Lambda\colon J^rTM \to \bigoplus_{k=0}^{r} TM \otimes S^kT^*M \tag{19}$$

as follows. Write

$$I\colon J_0^rT\mathbb{R}^m \to \bigoplus_{k=0}^{r} T_0\mathbb{R}^m \otimes S^kT_0^*\mathbb{R}^m$$

for the standard identification. Let φ be a Λ-normal coordinate system on M with center x and $B \in J_x^rTM$. We define

$$\psi_\Lambda(B) = \bigoplus_{k=0}^{r}(T_0\varphi^{-1} \otimes S^kT_0^{*-1}\varphi^{-1})\big(I\big(J^rT\varphi(B)\big)\big)\,. \tag{20}$$

Since the identification I is G_m^1-equivariant, (20) is a correct definition.

Proposition 12. *Let $D\colon C^\infty Q_\tau P^1M \to C^\infty QP^rM$ be a natural operator. Then there exist uniquely determined natural operators*

$$A_k\colon C^\infty Q_\tau P^1M \to C^\infty(TM \otimes S^kT^*M \otimes T^*M)\,,$$

$k = 0, \dots, r$, such that $A_0 = 0$, $A_1 = 0$ and

$$D(\Lambda) = E_r(\Lambda) + \big(0, 0, A_2(\Lambda), \dots, A_r(\Lambda)\big)$$

in the sense of the identification (19)*.*

Proof. The difference $D(\Lambda) - E_r(\Lambda)$ is decomposed into $r+1$ natural operators by (19). The natural operators A_0 and A_1 vanish according to 25.3 and Lemma 33.4 in [18]. □

Now Proposition 3 yields directly

Proposition 13. *Let $D\colon C^\infty Q_\tau P^1M \to C^\infty Q_\tau P^rM$ be a natural operator. Then there exist uniquely determined natural operators*

$$A_k\colon C^\infty Q_\tau P^1M \to C^\infty\big(TM \otimes S^{k+1}T^*M\big)\,,$$

$k = 0, \dots, r$, such that $A_0 = 0$, $A_1 = 0$ and

$$D(\Lambda) = E_r(\Lambda) + \big(0, 0, A_2(\Lambda), \dots, A_r(\Lambda)\big)\,.$$

The natural operators $A_2, \dots, A_r$ in Proposition 12 or Proposition 13 can be prescribed arbitrarily.

According to Lemma 33.4 of [18], all natural operators $C^\infty Q_\tau P^1 M \to C^\infty\big(TM \otimes \bigotimes^k T^*M\big)$ are $\mathbb{R}$-linearly generated by

– the curvature tensor and its covariant derivatives,

– constructing tensor products (including tensor products with invariant tensors) and contractions.

In the case of some prescribed symmetries in the covariant part we add the corresponding symmetrizations of the operators in question.

Example 1. Write $R^i_{jkl} = -R^i_{jlk}$ for the curvature tensor of Λ. If we look for all natural operators $D\colon C^\infty Q_\tau P^1 M \to C^\infty Q_\tau P^2 M$, we have to determine all natural operators $C^\infty Q_\tau P^1 M \to C^\infty(TM \otimes S^3 T^* M)$. By the above mentioned procedure, all these operators are the constant multiples of $\delta^m_{(j}\mathbb{R}^i_{kl)m}$. Hence all D's form the one-parameter family

$$D(\Lambda) = E_2(\Lambda) + c(\delta^m_{(j} R^i_{kl)m}), \qquad c \in \mathbb{R}.$$

Example 2. J. Janyška and the author determined all natural operators $C^\infty Q_\tau P^1 M \to C^\infty Q P^2 M$ by using a direct approach, [7], [12]. The list of them is rather long. Using Proposition 12, we can interpret that list in a very clear geometric way.

9. Semiholonomic 2-jets. Some aspects of our problems are properly related with the theory of semiholonomic 2-jets. First we describe the general ideas, [3], [23].

Consider a fibered manifold $p\colon Y \to M$. Its second nonholonomic prolongation $\tilde{J}^2 Y$ is defined by the iteration

$$\tilde{J}^2 Y = J^1(J^1 Y \to M).$$

If x^i, y^p are some fiber coordinates on Y, the induced coordinates on $J^1 Y$ are $y^p_r = \partial_i y^p(x)$ and the coordinates further induced on $\tilde{J}^2 Y$ are

$$y^p_{0i} = \partial_i y^p(x) \quad \text{and} \quad y^p_{ij} = \partial_j y^p_i(x).$$

There are two canonical projections $\tilde{J}^2 Y \to J^1 Y$, namely the target jet projection $\beta_1\colon \tilde{J}^2 Y \to J^1 Y$ and the jet prolongation $J^1\beta\colon \tilde{J}^2 Y \to J^1 Y$ of the target jet projection $\beta\colon J^1 Y \to Y$. The second semiholonomic prolongation $\bar{J}^2 Y$ is the set of all $A \in \tilde{J}^2 Y$ satisfying

$$\beta_1(A) = (J^1\beta)(A).$$

In coordinates, this condition means

$$y^p_{0i} = y^p_i. \tag{21}$$

The injection $J^2Y \hookrightarrow \tilde J^2 Y$ is defined by

$$j^2_x s \mapsto j^1_x(j^1 s)\,.$$

So the subset $J^2Y \subset \tilde J^2Y$ is characterized by

$$y^p_i = y^p_{0i} \quad\text{and}\quad y^p_{ij} = y^p_{ji}\,. \tag{22}$$

Hence $J^2Y \subset \bar J^2 Y$. According to the general theory, both $\beta_1\colon \tilde J^2Y \to J^1Y$ and $J^1\beta\colon \tilde J^2 Y \to J^1 Y$ are affine bundles.

For two manifolds M and N, the space $\tilde J^2(M,N)$ or $\bar J^2(M,N)$ of nonholonomic or semiholonomic 2-jets of M into N is the second nonholonomic or semiholonomic prolongation of the product fibered manifold $M\times N\to M$, respectively. C. Ehresmann introduced the composition of nonholonomic jets, [3]. Consider another manifold Q and $A\in \tilde J^2_x(M,N)_y$, $B\in \tilde J^2_y(N,Q)_z$. So $A = j^1_x\varphi$ and $B = j^1_y\psi$, where $\varphi\colon M\to J^1(M,N)$ and $\psi\colon N\to J^1(N,Q)$ are sections of the source jet projection α. Hence $\alpha\psi\big(\beta\varphi(u)\big) = \beta\varphi(u)$, $u\in M$, so that the composition of 1-jets $\psi\big(\beta\varphi(u)\big)$ and $\varphi(u)$ is defined. Then we set

$$B\circ A = j^1_x\big[\psi\big(\beta\varphi(u)\big)\circ\varphi(u)\big] \in \tilde J^2_x(M,Q)_z\,. \tag{23}$$

Let z^a be some local coordinates on Q, z^a_p, z^a_{0p}, z^a_{pq} be the induced coordinates on $\tilde J^2(N,Q)$ and w^a_i, w^a_{0i}, w^a_{ij} be the induced coordinates on $\tilde J^2(M,Q)$. Evaluating (23), we obtain the coordinate formula for the composition of nonholonomic 2-jets

$$w^a_i = z^a_p y^p_i\,,\quad w^a_{0i} = z^a_{0p}y^p_{0i}\,,\quad w^a_{ij} = z^a_{pq}y^p_i y^q_{0j} + z^a_p y^p_{ij}\,. \tag{24}$$

Clearly, the composition of two semiholonomic or holonomic 2-jets is semiholonomic or holonomic as well.

A jet $A\in\tilde J^2_x(M,N)_y$ is called regular, if there exists $B\in \tilde J^2_y(N,M)_x$ such that $B\circ A = j^2_x \operatorname{id}_M$. By (24), A is regular iff both $\beta_1(A)$ and $(J^1\beta)(A)$ are regular. In coordinates this means that both y^p_i and y^p_{0i} are regular matrices. If $\dim M = \dim N$, then $B\circ A = j^2_x\operatorname{id}_M$ implies $A\circ B = j^2_y\operatorname{id}_N$. In this case, regular is equivalent to invertible.

Every $\varphi(u)$ defines a linear map $T_uM\to TN$, $\eta^p = y^p_i(u)\xi^i$. This yields a local map $TM\to TN$, whose tangent map at each point of T_xM is determined by $j^1_x\varphi$. So the nonholonomic 2-jet $A = j^1_x\varphi(u)\in \tilde J^2_x(M,N)_y$ can be interpreted as a map $(TTM)_x\to(TTN)_y$ of the form

$$\eta^p = y^p_i\xi^i\,,\quad dy^p = y^p_{0i}dx^i\,,\quad d\eta^p = y^p_{ij}\xi^i dx^j + y^p_i d\xi^i\,. \tag{25}$$

Consider the canonical involution ι of the iterated tangent functor, $\iota_M(\xi^i, dx^i, d\xi^i) = (dx^i,\xi^i,d\xi^i)$, and $A\in\bar J^2_x(M,N)_y$ in the form (25) with $y^p_{0i} = y^p_i$. Then the map $\iota_N \circ A \circ \iota_M$ is of the form

$$\eta^p = y^p_i \xi^i\,, \quad dy^p = y^p_i dx^i\,, \quad d\eta^p = y^p_{ji}\xi^i dx^j + y^p_i d\xi^i\,. \tag{26}$$

This map corresponds to another semiholonomic 2-jet

$$\varkappa(A) \in \bar J^2_x(M,N)_y\,, \quad \varkappa(y^p_i, y^p_{ij}) = (y^p_i, y^p_{ji})\,. \tag{27}$$

Definition 2. The map $\varkappa$ is called the canonical involution of semiholonomic 2-jets.

Since $\bar J^2(M,N) \to J^1(M,N)$ is an affine bundle, A and $\varkappa(A)$ determine a tensor

$$\Delta(A) = A - \varkappa(A) \in T_y(N) \otimes \Lambda^2 T^*_x M \tag{28}$$

called the difference tensor of semiholonomic 2-jet A. (J. Pradines uses the name “dissymétrie”, [30].) Clearly, A is holonomic, iff $\Delta(A) = 0$.

Example 3. We present the first remarkable application of this concept. Consider a general connection $\Gamma\colon Y \to J^1Y$ on an arbitrary fibered manifold $Y \to M$, [18]. If Γ is viewed as a morphism over M, we can construct $J^1\Gamma\colon J^1Y \to \tilde J^2 Y$. Clearly, the values of the composition $\Gamma' = J^1\Gamma \circ \Gamma$ lie in $\bar J^2 Y$. The difference tensor

$$\Delta \circ \Gamma'\colon Y \to VY \otimes \Lambda^2 T^* M$$

coincides with the curvature of Γ.

The second order semiholonomic frame bundle $\bar P^2 M$ of M is defined by $\bar P^2 M = \operatorname{reg} \bar J^2_0(\mathbb{R}^m, M)$. This is a principal bundle $\bar P^2 M(M, \bar G^2_m)$, where $\bar G^2_m = \operatorname{inv} \bar J^2_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is the second order semiholonomic jet group in dimension m. The inclusion $G^2_m \subset \bar G^2_m$ defines an injection (denoted by the same symbol) $l_1\colon G^1_m \to \bar G^2_m$. One verifies easily that (8) with $r = 1$ defines an identification $i^1_M\colon J^1P^1M \approx \bar P^2 M$. Consider a principal connection $\Gamma\colon P^1M \to J^1P^1M$ with the coordinate expression

$$x^i_{j,k} = \Gamma^i_{lk} x^l_j\,. \tag{29}$$

Using (8), we verify directly the following result by P. Libermann, [23].

Proposition 14. *The rule $\Gamma \mapsto i^1_M \circ \Gamma$ defines a bijection between the connections on P^1M and the reductions of $\bar P^2 M$ to the subgroup $l_1(G^1_m) \subset \bar G^2_m$.*

It is remarkable that the canonical involution $\varkappa$ yields a simple construction of the connection $\tilde\Gamma$ conjugate to Γ. Indeed, we have

$$i^1_M \circ \tilde\Gamma = \varkappa \circ (i^1_M \circ \Gamma)\,, \tag{30}$$

i.e. $\varkappa$ transforms the reduction determined by Γ into the reduction determined by $\tilde\Gamma$.

Example 4. Another interesting application of the concept of difference tensor is in the theory of G-structures. We recall that a (first order) G-structure on M is a reduction P of P^1M to a subgroup $G \subset G^1_m$. Then $J^1P \subset J^1P^1M \approx \bar{P}^2M$ and Δ suggests a very conceptual way to the construction of the structure function of P, [19], [23]. In particular, this approach clarifies, in an instructive way, the difference between the prolongability and the flatness of P, [19].

Finally we remark that the theory of the covariant differentation with respect to connections on P^2M is systematically developed in [6].

10. W^rP as a generalization of P^rM. Consider a principal bundle $\pi\colon P \to M$ with structure group G, $\dim M = m$. Its r-th order principal prolongation W^rP is the bundle of all r-jets $j^r_{(0,e)}\varphi$ of local principal bundle isomorphisms

(31) $$\varphi\colon \mathbb{R}^m \times G \to P\,, \quad 0 \in \mathbb{R}^m, \quad e = \text{the unit of } G.$$

This is a principal bundle over M with structure group $W^r_mG := W^r_0(\mathbb{R}^m \times G)$, whose action on W^rP is given by the jet composition, [2], [18].

Given $A = j^r_0 f \in T^r_m P$, we write

$$\pi A = j^r_0(\pi \circ f) \in T^r_m M\,.$$

Further, we introduce

$$\operatorname{reg}_\pi T^r_m P = \{A \in T^r_m P; \quad \pi A \in \operatorname{reg} T^r_m M\}\,.$$

Clearly, the local $\mathcal{PB}$-isomorphism (31) is determined by its restriction $\varphi \mid \mathbb{R}^m \times \{e\}\colon \mathbb{R}^m \to P$. Hence

(32) $$W^rP = \operatorname{reg}_\pi T^r_m P\,.$$

Let $\bar{P}(\bar{M}, G)$ be another principal G-bundle, $m = \dim \bar{M}$. For every local principal bundle isomorphism $f\colon P \to \bar{P}$, $T^r_m f\colon T^r_m P \to T^r_m \bar{P}$ restricts and corestricts into a map $\operatorname{reg}_\pi T^r_m P \to \operatorname{reg}_\pi T^r_m \bar{P}$. This defines $W^rf\colon W^rP \to W^r\bar{P}$.

If $A \in \operatorname{reg}_\pi T^r_m P$, then $\pi A \in T^r_m M$ is invertible, so that $A \circ (\pi A)^{-1}$ satisfies $\pi\big(A \circ (\pi A)^{-1}\big) = j^r_x \operatorname{id}_M$, $x = \pi\big(f(0)\big)$. This implies $A \circ (\pi A)^{-1} \in J^rP$. Hence

(33) $$W^rP = P^rM \times_M J^rP\,.$$

Clearly, W^rf is identified with $P^r\underline{f} \times_{\underline{f}} J^rf$, where $\underline{f}\colon M \to \bar{M}$ is the base map of f. In particular, the structure group

(34) $$W^r_0(\mathbb{R}^m \times G) =: W^r_mG = G^r_m \rtimes T^r_mG$$

is the group semidirect product with the group composition

$$(g_1, C_1)(g_2, C_2) = \big(g_1 \circ g_2, (C_1 \circ g_2) \bullet C_2\big)\,,$$

where $\bullet$ denotes the induced group composition in $T^r_m G$, [18]. The first product projection $W^r P \to P^r M$ is a principal bundle morphism with the associated group homomorphism $W^r_m G \to G^r_m$ determined by (34).

$W^r P$ is a fundamental structure for the gauge theories of mathematical physics, [5]. In differential geometry, the main role of $W^r P$ is based on the fact that for every associated bundle $P[S]$, where S is a left G-space, the r-th jet prolongation $J^r\big(P[S]\big)$ is a fiber bundle associated to $W^r P$, [18]. Further, we have a canonical injection $P^r M \hookrightarrow W^1 P^{r-1} M$, $j^r_0 f \mapsto j^1_{(0,e_{r-1})} P^{r-1} f$, $f \colon \mathbb{R}^m \to M$. So $W^1 P$ can play the role of a suitable recurrence model for several geometric problems, [18]. In particular, the reductions of the principal bundle $W^1 P$ are called generalized G-structures, [10]. Several properties of higher order G-structures are well reflected in the framework of this more general theory.

If $G = \{e\}$ is the one-element group, then $M \times \{e\}$ is identified with M and $W^r(M \times \{e\}) = P^r M$. Hence many properties of $W^r P$ can be viewed as a generalization of the case of $P^r M$. In particular, we have the canonical one-form $\varphi_r \colon TP^r M \to \mathbb{R}^m \times \mathfrak{g}^{r-1}_m$ on $P^r M$. On $W^r P$, we introduce analogously a canonical one-form $\theta_r \colon TW^r P \to \mathbb{R}^m \times \mathfrak{w}^{r-1}_m G = T_{(0,E_{r-1})} W^{r-1}(\mathbb{R}^m \times G)$, E_{r-1} = the unit of $W^{r-1}_m G$, $\mathfrak{w}^{r-1}_m G = \mathrm{Lie}\,(W^{r-1}_m G)$. Consider $u = j_{(0,e)}\psi \in W^r P$ and write $u_1 = \pi^r_{r-1}(u) \in W^{r-1} P$, where π^r_{r-1} is the jet projection. The tangent map

$$\tilde{u} = T_{(0,E_{r-1})} W^{r-1}\psi \colon \mathbb{R}^m \times \mathfrak{w}^{r-1}_m G \to T_{u_1} W^{r-1} P$$

is a linear isomorphism depending on u only. For every $Z \in T_u W^r P$, we define

$$\theta_r(Z) = \tilde{u}^{-1}\big(T\pi_{r-1}(Z)\big)\,.$$

Clearly, the following diagram commutes

$$\begin{array}{ccc} TW^r P & \xrightarrow{\ \theta_r\ } & \mathbb{R}^m \times \mathfrak{w}^{r-1}_m G \\ \downarrow & & \downarrow \\ TP^r M & \xrightarrow{\ \varphi_r\ } & \mathbb{R}^m \times \mathfrak{g}^{r-1}_m \end{array}$$

Analogously to Section 2, we introduced in [9]

Definition 3. The torsion of a connection Φ on $W^r P$ is the covariant exterior differential $D_\Phi \theta_r$.

The Lie algebroid $LW^r P$ of $W^r P$ coincides with the r-th jet prolongation $J^r(LP \to M)$, [17], [22]. Let $\varphi \colon TM \to J^r LP$ be the algebroid form of Φ. Analogously to the case of J^rTM, we have the truncated bracket

$$[\![\,,\,]\!]_{r-1}\colon J^rLP \times_M J^rLP \to J^{r-1}LP\,.$$

The torsion $\tau\varphi$ of φ can be introduced as a morphism

$$\tau\varphi\colon TM \times_M TM \to J^{r-1}LP$$

defined by

$$(\tau\varphi)(Z_1, Z_2) = [\![\varphi(Z_1), \varphi(Z_2)]\!]_{r-1}\,, \quad (Z_1, Z_2) \in TM \times_M TM\,.$$

In [17] it is deduced that $D_\Phi\theta_r$ and $\tau\varphi$ are naturally equivalent.

The r-jets $j^r_0\hat{g}$, $g \in G$, of the constant maps $\hat{g}\colon \mathbb{R}^m \to G$, $x \mapsto g$, define an injection $\nu^r_m\colon G \to T^r_mG$. Clearly, the direct group product $l_{r-1}(G^1_m) \times \nu^r_m(G)$ is a subgroup of W^r_mG. The following assertion, that is an analogy of Proposition 5, is proved in [17].

Proposition 15. *The torsion-free connections on W^rP are in bijection with the reductions of $W^{r+1}P$ to the subgroup $l_r(G^1_m) \times \nu^{r+1}_m(G) \subset W^{r+1}_mG$.*

11. Connections on W^1P. We are going to discuss the connections on W^1P in more details. By (33),

$$W^1P = P^1M \times_M J^1P\,. \tag{35}$$

Write $p_1\colon W^1P \to P^1M$ and $p_2\colon W^1P \to J^1P$ for the product projections. Since $p_1\colon W^1P \to P^1M$ and the target jet projection $\beta\colon W^1P \to P$ are principal bundle morphisms, every connection

$$\Phi\colon W^1P \to J^1(W^1P) = J^1P^1M \times_M \tilde{J}^2P \tag{36}$$

induces a pair of connections $p_{1*}\Phi$ on P^1M and $\beta_*\Phi$ on P.

Conversely, consider two connections $\Gamma\colon P \to J^1P$ and $\Lambda\colon P^1M \to J^1P^1M$. Define

$$W^1P \supset R(\Gamma) = \big\{\big(u, \Gamma(v)\big); (u,v) \in P^1M \times_M P\big\}\,. \tag{37}$$

Using the action ϱ of G on P from Section 3, one finds easily that $R(\Gamma)$ is a reduction of W^1P to the subgroup $G^1_m \times \nu^1_m(G) \subset W^1_mG$. Since (37) identifies $R(\Gamma)$ with $P^1M \times_M P$, the product connection $\Lambda \times \Gamma$ on $P^1M \times_M P$ is identified with a connection on $R(\Gamma)$ and the latter connection is uniquely extended into a connection $p(\Gamma, \Lambda)$ on W^1P. Clearly, $\beta_*p(\Gamma,\Lambda) = \Gamma$ and $p_{1*}p(\Gamma,\Lambda) = \Lambda$.

Write L^0P for the adjoint bundle of P. (Our notation is motivated by the fact that L^0P is the subset of the Lie algebroid LP of all elements A satisfying $q(A) = 0$.) The projections β and p_1 give rise to projections $L^0W^1P \otimes T^*M \to L^0P \otimes T^*M$ and $L^0W^1P \otimes T^*M \to L^0P^1M \otimes T^*M$. The common kernel of these projections is $L^0P \otimes \bigotimes^2 T^*M$, [12].

Proposition 16. *Connections on W^1P are in bijection with triples (Γ, Λ, D), where $\Gamma \in C^\infty(QP)$, $\Lambda \in C^\infty(QP^1M)$ and $D \in C^\infty(L^0P \otimes \overset{2}{\bigotimes} T^*M)$.*

Proof. For $\Phi \in C^\infty(QW^1P)$ we set $\Gamma = \beta_*\Phi$, $\Lambda = p_{1*}\Phi$ and $D = \Phi - p(\beta_*\Phi, p_{1*}\Phi)$. □

The factor $T^*M \otimes T^*M$ gives rise to an exchange map ex: $L^0P \otimes \overset{2}{\bigotimes} T^*M \to L^0P \otimes \overset{2}{\bigotimes} T^*M$. Thus, if we replace Λ by the classical conjugate connection $\tilde{\Lambda}$ and D by ex $\circ\, D$, we obtain a connection $\tilde{\Phi}$ said to be conjugate to Φ. In Section 13 we present a more geometric construction of $\tilde{\Phi}$ by using the canonical involution of semiholonomic 2-jets.

There is another construction transforming the pair (Γ, Λ) into a connection on W^1P. It is based on the general idea of flow prolongation of connections, [18]. Consider Γ in the lifting form

$$\Gamma \colon P \times_M TM \to TP\,.$$

For every vector field X on M, we first construct its Γ-lift $\Gamma X \colon P \to TP$ and then the flow prolongation $\mathcal{W}^1(\Gamma X) \colon W^1P \to TW^1P$. This defines a map

$$\mathcal{W}^1\Gamma \colon W^1P \times_M J^1TM \to TW^1P\,.$$

If we add Λ in its algebroid form $TM \to J^1TM$, we obtain the lifting map

$$\mathcal{W}^1(\Gamma, \Lambda) \colon W^1P \times_M TM \to TW^1P$$

of a principal connection on W^1P. In [20] we deduced $\beta_*\mathcal{W}^1(\Gamma, \Lambda) = \Gamma$, $p_{1*}\mathcal{W}^1(\Gamma, \Lambda) = \tilde{\Lambda}$. So the difference $p(\Gamma, \Lambda) - \mathcal{W}^1(\Gamma, \tilde{\Lambda})$ is a section of $L^0P \otimes \overset{2}{\bigotimes} T^*M$. We recall that the curvature $C(\Gamma)$ of Γ is a section of $L^0P \otimes \Lambda^2 T^*M$. In [20], we proved

Proposition 17. *We have $p(\Gamma, \Lambda) - \mathcal{W}^1(\Gamma, \tilde{\Lambda}) = C(\Gamma)$.*

12. Gauge-natural operators on connections. Analogously to Section 8 one can pose the question of finding all geometric operators transforming a pair (Γ, Λ) of a connection Γ on P and a connection Λ on P^1M into a connection on W^1P. The precise meaning of "geometric" is "gauge-natural" in the sense of [2]. Roughly speaking, when passing from the classical natural operators to the gauge-natural ones, we meet the higher order principal prolongations W^rP in the former role of the higher order frame bundles P^rM, see [18] for a complete theory.

In [20] we deduced the following list of all gauge-natural operators $A\colon C^\infty(QP)\times C^\infty(Q_\tau P^1M)\to C^\infty(QW^1P)$. (The assumption Λ is torsion-free is of technical character. If we replace $Q_\tau P^1M$ by QP^1M, the list will be much longer with many further terms of less geometric interest.) First of all one proves that the underlying connections of $A(\Gamma,\Lambda)$ are $\beta_*A(\Gamma,\Lambda)=\Gamma$ and $p_{1*}A(\Gamma,\Lambda)=\Lambda$. So the difference $A(\Gamma,\Lambda)-p(\Gamma,\Lambda)$ is a section of $L^0P\otimes\overset{2}{\bigotimes}T^*M$.

We already know that the curvature $C(\Gamma)$ is a section of that bundle. Let $Z\subset\mathrm{Lin}(\mathfrak{g},\mathfrak{g})$ be the subspace of all linear maps commuting with the adjoint action of G. Since every $z\in Z$ is an equivariant map between the standard fibers, it induces a vector bundle morphism $z_P\colon L^0P\to L^0P$. Hence one can construct the modified curvature operator $C(\Gamma)(z)=(z_P\otimes\mathrm{id}\,)\circ C(\Gamma)$. On the other hand, by Example 28.7 of [18] all natural operators $C^\infty(Q_\tau P^1M)\to C^\infty(T^*M\otimes T^*M)$ are linearly generated by the contractions $R_1(\Lambda)=(R^k_{kij})$ and $R_2(\Lambda)=(R^k_{ikj})$ of the curvature tensor (R^i_{jkl}) of Λ. Let $S\subset\mathfrak{g}$ be the subspace of all vectors invariant with respect to the adjoint action. Since every $B\in S$ is an invariant element of the standard fiber, it determines a section B_P of L^0P. Our result from [20] reads

Proposition 18. *All gauge-natural operators $C^\infty(QP)\times C^\infty(Q_\tau P^1M)\to C^\infty(QW^1P)$ are of the form*

$$p(\Gamma,\Lambda)+C(\Gamma)(z)+B_{1P}\otimes R_1(\Lambda)+B_{2P}\otimes R_2(\Lambda)$$

for all $z\in Z$ and all $B_1,B_2\in S$.

We underline that there exist many interesting open problems concerning the gauge-natural operators related with connections on W^rP.

13. Connections on W^1P as reductions of $\bar W^2P$. Finally we outline how the semiholonomic 2-jets can be used in the theory of connections on W^1P. In general, the bundle of nonholonomic $(n,2)$-velocities on a manifold M is defined by

$$\tilde T^2_nM=\tilde J^2_0(\mathbb{R}^n,M)\,.$$

We shall frequently use a natural identification

$$\tilde T^2_nM\approx T^1_n(T^1_nM)\,. \tag{38}$$

Write $t_u\colon\mathbb{R}^n\to\mathbb{R}^n$ for the translation $x\mapsto x+u$. If $\psi\colon\mathbb{R}^n\to J^1(\mathbb{R}^n,M)$ is a section, then $u\mapsto\psi(u)\circ j^1_0t_u$ is a map $\mathbb{R}^n\to T^1_nM$. Passing to 1-jets defines (38). One verifies easily that (38) identifies $\mathrm{reg}\,\tilde T^2_nM$ with $\mathrm{reg}\,T^1_n(\mathrm{reg}\,T^1_nM)$.

The second order nonholonomic frame bundle of M is defined by

$$\tilde{P}^2 M = \operatorname{inv} \tilde{J}^2_0(\mathbb{R}^m, M)\,.$$

This is a principal bundle over M with structure group

$$\tilde{G}^2_m = \operatorname{inv} \tilde{J}^2_0(\mathbb{R}^m, \mathbb{R}^m)_0\,,$$

the right action of which on $\tilde{P}^2 M$ is determined by the composition of nonholonomic 2-jets. On the other hand,

$$W^1(P^1M) = P^1M \times_M J^1P^1M\,.$$

Using (32) and (38), we obtain

$$\tilde{P}^2 M = W^1(P^1M)\,. \tag{39}$$

In particular, $\tilde{G}^2_m = G^1_m \rtimes T^1_m G^1_m$.

Further we introduce the second nonholonomic prolongation of $P(M,G)$ by the iteration

$$\tilde{W}^2 P = W^1(W^1 P)\,. \tag{40}$$

This is a principal bundle over M with structure group $\tilde{W}^2_m G := W^1_m(W^1_m G)$. We have

$$W^1(W^1P) = P^1M \times_M J^1P^1M \times_M \tilde{J}^2 P = \tilde{P}^2 M \times_M \tilde{J}^2 P$$

and

$$W^1_m(W^1_m G) = G^1_m \times T^1_m G^1_m \times T^1_m T^1_m G \approx \tilde{G}^2_m \rtimes \tilde{T}^2_m G\,,$$

where the group semidirect product has an analogous meaning to (34). In the semiholonomic case, we have

$$\tilde{W}^2 P \supset \bar{W}^2 P := \bar{P}^2 M \times_M \bar{J}^2 P\,. \tag{41}$$

The structure group of $\bar{W}^2 P$ is $\bar{W}^2_m G = \bar{G}^2_m \rtimes \bar{T}^2_m G$.

Consider a connection $\Phi\colon W^1P \to J^1(W^1P) = J^1P^1M \times_M \tilde{J}^2 P$. Using the identification $J^1P^1M \approx \bar{P}^2 M$ from Section 9 and the inclusion $\bar{J}^2 P \subset \tilde{J}^2 P$, we obtain an inclusion $\bar{W}^2 P \subset J^1 W^1 P$. Let Γ and Λ be the underlying connections of Φ. In Section 11 we constructed the reduction $R(\Gamma) \subset W^1 P$ to the subgroup $G^1_m \times \nu^1_m(G)$. One verifies easily that $\Phi\big(R(\Gamma)\big) \subset \bar{W}^2 P$ is a reduction to the subgroup $l_1(G^1_m) \times \nu^2_m(G) \subset G^2_m \rtimes T^2_m G \subset \bar{G}^2_m \rtimes \bar{T}^2_m G$. The following assertion generalizes the result by P. Libermann mentioned in Section 9.

Proposition 19. *The connections on W^1P are in bijection with the reductions of $\bar{W}^2 P$ to the subgroup $l_1(G^1_m) \times \nu^2_m(G) \subset \bar{G}^2_m \rtimes \bar{T}^2_m G$.*

Proof. Let Q be a reduction of $\bar{W}^2P$ to $l_1(G^1_m) \times \nu^2_m(G)$. Hence its projection Q_1 into W^1P is a reduction to the subgroup $G^1_m \times \nu^1_m(G)$. Then Q can be interpreted as a map (denoted by the same symbol) $Q\colon Q_1 \to J^1W^1P$. This map is equivariant, so that Q can be uniquely extended into a connection on W^1P. □

Now we can construct the connection $\tilde{\Phi}$ conjugate to Φ by using the canonical involution $\varkappa$ of semiholonomic 2-jets. Using Proposition 3 of [20], one proves that if $\mathrm{red}(\Phi) \subset \bar{W}^2P$ is the reduction corresponding to Φ, then

$$\varkappa \circ \mathrm{red}(\Phi) = \mathrm{red}(\tilde{\Phi}) \tag{42}$$

is the reduction corresponding to $\tilde{\Phi}$.

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