ON SPECIAL TYPES OF SEMIHOLONOMIC 3-JETS

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Abstract. We start with some general ideas concerning the concept of special type of nonholonomic r-jets. Then we classify the special types of semiholonomic 3-jets.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [8].

1. Introduction

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps and $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms. Every two manifolds M and N determine the bundle $J^r(M,N) \to M \times N$ of all r-jets of M into N. In [8] we pointed out that J^r is a bundle functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$, $m = \dim M$. Indeed, every local diffeomorphism $f\colon M \to M'$ and every map $g\colon N \to N'$ induce a map

$$J^r(f,g)\colon J^r(M,N) \to J^r(M',N')$$

by the jet composition

$$J^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \quad X \in J^r_x(M,N)_y\,. \tag{1}$$

Clearly, $J^r(M, N_1 \times N_2) = J^r(M,N_1) \times_M J^r(M,N_2)$.

In [1], C. Ehresmann introduced the bundle $\tilde{J}^r(M,N) \to M \times N$ of nonholonomic r-jets of M into N, $J^r(M,N) \subset \tilde{J}^r(M,N)$, see also [5]. He defined a composition

$$X_2 \circ X_1 \in \tilde{J}^r_x(M,Q)_z \tag{2}$$

for every $X_1 \in \tilde{J}^r_x(M,N)_y$ and $X_2 \in \tilde{J}^r_y(N,Q)_z$, that is associative and generalizes the composition of the classical holonomic r-jets. Hence $\tilde{J}^r$ can be interpreted as a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$, if we set

$$\tilde{J}^r(f,g) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \quad X \in \tilde{J}^r_x(M,N)_y\,, \tag{3}$$

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with the composition of nonholonomic r-jets. Even in this case we have $\tilde J^r(M, N_1 \times N_2) = \tilde J^r(M, N_1) \times_M \tilde J^r(M, N_2)$.

The best known example of special type of nonholonomic r-jets are the bundles $\bar J^r(M, N)$ of semiholonomic r-jets

$$J^r(M, N) \subset \bar J^r(M, N) \subset \tilde J^r(M, N)\,,$$

[1], [5], [9]. There is a simple description of $\bar J^r(V, W)$ in the case of two vector spaces V, W, [1]. Analogously to the classical formula

$$J^r(V, W) = V \oplus W \otimes \Big(\sum_{i=0}^{r} S^i V^*\Big) \tag{4}$$

with symmetric tensor powers of V^*, we have

$$\bar J^r(V, W) = V \oplus W \otimes \Big(\sum_{i=0}^{r} \overset{i}{\otimes} V^*\Big) \tag{5}$$

with arbitrary tensor powers of V^*. The composition of two semiholonomic r-jets is semiholonomic as well. Further, $\bar J^r(M, N_1 \times N_2) = \bar J^r(M, N_1) \times_M \bar J^r(M, N_2)$. We denote by $\pi^r_s \colon \bar J^r(M, N) \to \bar J^s(M, N)$, $s < r$, the canonical projection, [1].

We have been interested in the general concept of special type of nonholonomic r-jets. In our first attempt, [3], we started from the description of all bundle functors on the category $\mathcal{M}f_m \times \mathcal{M}f$ preserving product in the second factor, [7], [5]. In general, a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ is said to preserve products in the second factor, if

$$F(M, N_1 \times N_2) = F(M, N_1) \times_M F(M, N_2)\,.$$

Further, F is said to be of order r in the first factor, if for every two local diffeomorphisms f_1, $f_2 \colon M_1 \to M_2$ and every $g \colon N_1 \to N_2$, $j^r_x f_1 = j^r_x f_2$ implies

$$F(f_1, g) \mid F_x(M_1, N_1) = F(f_2, g) \mid F_x(M_1, N_1)\,,$$

where $F_x(M_1, N_1)$ means the fiber of $F(M_1, N_1)$ over $x \in M_1$. Such functors are identified with pairs (A, H), where A is a Weil algebra and $H \colon G^r_m \to \operatorname{Aut} A$ is a group homomorphism of the r-th jet group G^r_m in dimension m into the group $\operatorname{Aut} A$ of all algebra automorphisms of A. Then $F(M, N)$ is the associated bundle $P^r M[T^A N, H_N]$, where $P^r M$ is the r-th order frame bundle of M and H_N is the induced action of G^r_m on $T^A N$. We have $F(f, g) = P^r f[T^A g]$.

In the special case $F = J^r$, the Weil algebra is $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$, we have $\operatorname{Aut} \mathbb{D}^r_m \approx G^r_m$ and $H = \mathrm{id}_{G^r_m}$. This yields a classical formula $J^r(M, N) = P^r M[T^r_m N]$. In the case $F = \tilde J^r$, the Weil algebra is $\tilde{\mathbb{D}}^r_m = \tilde J^r_0(\mathbb{R}^m, \mathbb{R})$, $T^{\tilde{\mathbb{D}}^r_m} N = \tilde T^r_m N = \tilde J^r_0(\mathbb{R}^m, N)$ is the bundle of nonholonomic

(m,r)-velocities over N, the jet composition defines an action of G^r_m on $\tilde{\mathbb{D}}^r_m$ and $\tilde{J}^r(M,N) = P^rM[\tilde{T}^r_m N]$.

In our first approach, [3], we considered a G^r_m-invariant Weil algebra Φ, $\mathbb{D}^r_m \subset \Phi \subset \tilde{\mathbb{D}}^r_m$, and we defined an r-th order jet functor on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$F(M,N) = P^rM[T^\Phi N, i^\Phi_N]\,, \quad F(f,g) = P^rf[T^\Phi g]\,, \tag{6}$$

where i^Φ is the action of G^r_m on Φ. Clearly,

$$J^r(M,N) \subset F(M,N) \subset \tilde{J}^r(M,N)\,. \tag{7}$$

Conversely, if F is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ satisfying (7) and preserving products in the second factor, then F is determined by a Weil algebra Φ of the above type, [3].

Using the Weil algebra technique, [4], we deduced that the only nonholonomic 2-jet functors on $\mathcal{M}f_m \times \mathcal{M}f$ are J^2, $\bar{J}^2$ and $\tilde{J}^2$.

However, this model does not includes the composition of jets. That is why we have recently introduced the general concept of nonholonomic r-jet category C, [6]. In Section 2 of the present paper, we describe C in terms of its skeleton. Then we deduce some algebraic properties of the algebra $\tilde{\mathbb{D}}^r_m$ and we characterize C in terms of the induced sequence $\mathbb{D}^C_m \subset \tilde{\mathbb{D}}^r_m$ of Weil algebras. Our above mentioned result from [4] implies directly that the only nonholonomic 2-jet categories are J^2, $\bar{J}^2$ and $\tilde{J}^2$, see Example 2 below. However, there are so many nonholonomic 3-jet categories that we do not find reasonable to classify all of them without further reasons. So we restrict ourselves to the semiholonomic 3-jet categories and we describe the generic case in Section 4.

2. Nonholonomic r-jet categories

We recall that $X \in \tilde{J}^r_x(M,N)_y$ is said to be regular, if there exists $Z \in \tilde{J}^r_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$, [6].

In [6], we introduced a nonholonomic r-jet category C as a rule transforming every pair (M,N) of manifolds into a fibered submanifold $C(M,N) \subset \tilde{J}^r(M,N)$ such that

(i) $J^r(M,N) \subset C(M,N)$ is a fibered submanifold,
(ii) if $X \in C_x(M,N)_y$ and $Z \in C_y(N,Q)_z$, then $Z \circ X \in C_x(M,Q)_z$,
(iii) if $X \in C_x(M,N)_y$ is regular in $\tilde{J}^r(M,N)$, then there exists $Z \in C_y(N,M)_x$ such that $Z \circ X = j^r_x \operatorname{id}_M$,
(iv) $C(M, N \times Q) = C(M,N) \times_M C(M,Q)$.

Analogously to the case of J^r, [8], we define $L^C_{m,n} = C_0(\mathbb{R}^m, \mathbb{R}^n)_0$ and

$$L^C = \bigcup_{m,n\in\mathbb{N}} L^C_{m,n}$$

is called the skeleton of C. Clearly, we can reconstruct C from L^C in the same way as in the case of J^r, [8]. We have a left action of $G^r_m \times G^r_n$ on $L^C_{m,n}$

$$(g_1, g_2)(X) = g_2 \circ X \circ g_1^{-1}\,, \quad g_1 \in G^r_m,\ g_2 \in G^r_n,\ X \in L^C_{m,n} \tag{8}$$

and $C(M,N)$ coincides with the associated bundle

$$C(M,N) = (P^r M \times P^r N)[L^C_{m,n}]\,. \tag{9}$$

We define $T^C_m N = C_0(\mathbb{R}^m, N)$. This gives rise to a product preserving bundle functor on $\mathcal{M}f$, so a Weil functor $T^{\mathbb{D}^C_m}$, $\mathbb{D}^r_m \subset \mathbb{D}^C_m$. Clearly, each $\mathbb{D}^C_m$ is a G^r_m-invariant Weil subalgebra of $\tilde{\mathbb{D}}^r_m$. We are going to clarify how C can be determined by such a sequence.

3. Some algebraic properties of $\tilde{\mathbb{D}}^r_m$

By the iteration theorem for Weil bundles, [5], we have

$$\tilde{\mathbb{D}}^r_m \approx \mathbb{D}^1_m \underbrace{\otimes \cdots \otimes}_{r\text{-times}} \mathbb{D}^1_m\,, \quad \mathbb{D}^1_m = \mathbb{R} \times \mathbb{R}^{m*}\,. \tag{10}$$

Write e^i_s, $i = 1, \dots, m$, $s = 1, \dots, r$ for the canonical basis of $\mathbb{R}^{m*}$ and $e^0_s = 1_s$ for the unit in the s-th component of (10). For a sequence $k_1, \dots, k_r$ of $0, 1, \dots, m$, we define

$$e^{k_1\dots k_r} = e^{k_1}_1 \otimes \cdots \otimes e^{k_r}_r\,. \tag{11}$$

This is a basis of the vector space $\tilde{\mathbb{D}}^r_m$, so that every $X \in \tilde{\mathbb{D}}^r_m$ is of the form $X = x_{k_1\dots k_r} e^{k_1\dots k_r}$. The multiplication in $\tilde{\mathbb{D}}^r_m$ is determined by

$$e^{k_1\dots k_r} e^{l_1\dots l_r} = e^{h_1\dots h_r}\,, \tag{12}$$

where $e^{h_1\dots h_r} = 0$ if $k_s \neq 0 \neq l_s$ for at least one s and $h_s = k_s + l_s$ otherwise.

Write $\langle k_1 \dots k_r\rangle = (i_1 \dots i_s)$, $s \le r$, for the subsequence of all nonzero indices and $|k_1 \dots k_r|$ for the set $\{i_1, \dots, i_s\}$. The semiholonomic subalgebra $\bar{\mathbb{D}}^r_m = \bar{J}^r_0(\mathbb{R}^m, \mathbb{R})$ is characterized by

$$x_{k_1\dots k_r} = x_{l_1\dots l_r} \quad \text{whenever} \quad \langle k_1 \dots k_r\rangle = \langle l_1 \dots l_r\rangle \tag{13}$$

and the holonomic subalgebra $\mathbb{D}^r_m$ satisfies

$$x_{k_1\dots k_r} = x_{l_1\dots l_r} \quad \text{whenever} \quad |k_1 \dots k_r| = |l_1 \dots l_r|\,. \tag{14}$$

In the holonomic case, a simple assertion is that the set of all Weil algebra homomorphisms $\mathrm{Hom}\,(\mathbb{D}^r_m, \mathbb{D}^r_n)$ coincides with $L^r_{n,m}$, [5]. This identification is a special case of the following construction.

Proposition 1. *For every $Z \in \tilde{L}^r_{n,m}$ the rule*

$$Z^h(X) = X \circ Z\,, \quad X \in \tilde{\mathbb{D}}^r_m \tag{15}$$

defines a Weil algebra homomorphism $Z^h \colon \tilde{\mathbb{D}}^r_m \to \tilde{\mathbb{D}}^r_n$.

Proof. A quick proof is based on a general result concerning Weil bundles, [5], [8]. Consider the bundle functors $\tilde{T}^r_m$ and $\tilde{T}^r_n$ on $\mathcal{M}f$. For $f \colon Q \to Q'$ and $X \in (\tilde{T}^r_m Q)_x$, we have $\tilde{T}^r_m f(X) = j^r_x f \circ X$. Since the composition of nonholonomic jets in associative, we have $(\tilde{T}^r_m f(X)) \circ Z = (j^r_x f) \circ X \circ Z = \tilde{T}^r_n f(X \circ Z)$, so that Z induces a natural transformation $\tilde{T}^r_m \to \tilde{T}^r_n$. These are determined by the algebra homomorphisms $\tilde{\mathbb{D}}^r_m \to \tilde{\mathbb{D}}^r_n$. □

Write $\tilde{\mathbb{D}}^r_m = \mathbb{R} \times \tilde{N}^r_m$, so that $\tilde{N}^r_m = \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R})_0$. Since $\tilde{J}^r$ preserves products in the second factor, we have $\tilde{L}^r_{m,n} = \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^n)_0 = (\tilde{N}^r_m)^n$. Analogously, $\bar{L}^r_{m,n} := \bar{J}^r_0(\mathbb{R}^m, \mathbb{R})_0 = (\bar{N}^r_m)^n$ and $L^r_m = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0 = (N^r_m)^n$ with $\bar{\mathbb{D}}^r_m = \mathbb{R} \times \bar{N}^r_m$ and $\mathbb{D}^r_m = \mathbb{R} \times N^r_m$.

Proposition 2. *We have* $\mathrm{Hom}\,(\bar{\mathbb{D}}^r_m, \bar{\mathbb{D}}^r_n) = \bar{L}^r_{n,m}$.

Proof. Consider an algebra homomorphism $\varphi \colon \bar{\mathbb{D}}^r_m \to \bar{\mathbb{D}}^r_n$. The algebraic generators of $\bar{N}^r_m$ are $e^i := e^{i0\ldots0} + \cdots + e^{0\ldots0i}$. Write $\varphi^i = \varphi(e^i) \in \bar{N}^r_n$, so that $\Phi := (\varphi^1, \ldots \varphi^m) \in \bar{L}^r_{n,m}$. Then the algebra homomorphism Φ^h coincides with φ on the algebraic generators, so that $\varphi = \Phi^h$. □

Example 1. Direct evaluation in the case $r = 2$ shows that $\tilde{L}^2_{m,n}$ is a proper subset of $\mathrm{Hom}\,(\tilde{\mathbb{D}}^2_n, \tilde{\mathbb{D}}^2_m)$ only. Indeed, if we consider the standard coordinate expressions $a = (a^p_{i0}, a^p_{0i}, a^p_{ij}) \in \tilde{L}^2_{m,n}$ and $b = (b^v_{p0}, b^v_{0p}, b^v_{pq}) \in \tilde{L}^2_{n,p}$ the composition $c = b \circ a = (c^v_{i0}, c^v_{0i}, c^v_{ij}) \in \tilde{L}^2_{m,p}$, i, $j = 1, \ldots, m$, p, $q = 1, \ldots, n$, $v = 1, \ldots, p$, is of the form

$$\begin{aligned} c^v_{i0} &= b^v_{p0} a^p_{i0}\,, \quad c^v_{0i} = b^v_{0p} a^p_{0i}\,, \\ c^v_{ij} &= b^v_{pq} a^p_{i0} a^q_{0j} + b^v_{p0} a^p_{ij}\,. \end{aligned} \tag{16}$$

Thus, for $x = (x_{i0}, x_{0i}, x_{ij}) \in \tilde{N}^2_m$ and $a \in \tilde{L}^2_{n,m}$, we have

$$a^h(x) = x \circ a = (x_{i0} a^i_{p0}, x_{0i} a^i_{0p}, x_{ij} a^i_{p0} a^j_{0q} + x_{i0} a^i_{pq})\,. \tag{17}$$

On the other hand, an algebra homomorphism $f\colon \tilde{\mathbb{D}}^2_m \to \tilde{\mathbb{D}}^2_n$ is determined by

$$
\begin{aligned}
f(e^{i0}) &= d^{i0}_{p0} e^{p0} + d^{i0}_{0p} e^{0p} + d^{i0}_{pq} e^{pq}\,,\\
f(e^{0i}) &= d^{0i}_{p0} e^{p0} + d^{0i}_{0p} e^{0p} + d^{0i}_{pq} e^{pq}\,.
\end{aligned}
\tag{18}
$$

Then

$$
f(e^{ij}) = f(e^{i0}e^{0j}) = (d^{i0}_{p0} d^{0j}_{0q} + d^{i0}_{0q} d^{0j}_{p0}) e^{pq}\,. \tag{19}
$$

By direct evaluation, we find $f(x)$ in the form

$$
\begin{aligned}
&(x_{i0} d^{i0}_{p0} + x_{0i} d^{0i}_{p0}) e^{p0} + (x_{i0} d^{i0}_{0p} + x_{0i} d^{0i}_{0p}) e^{0p}\\
&\qquad + \big[x_{i0} d^{i0}_{pq} + x_{0i} d^{0i}_{pq} + x_{ij}(d^{i0}_{p0} d^{0j}_{0q} + d^{i0}_{0q} d^{0j}_{p0})\big] e^{pq}\,.
\end{aligned}
\tag{20}
$$

Clearly, (20) reduces to (17) iff $d^{0i}_{p0} = 0$, $d^{i0}_{0p} = 0$, $d^{0i}_{pq} = 0$.

Consider the immersion $i_{m,n}\colon \mathbb{R}^m \to \mathbb{R}^{m+n}$, $x \mapsto (x,0)$, and the submersion $s_{m,n}\colon \mathbb{R}^{m,n} \to \mathbb{R}^m$, $(x_1,x_2) \mapsto x_1$, and write $I^r_{m,n} = j^r_0 i_{m,n}$. Since $s_{m,n} \circ i_{m,n} = \mathrm{id}_{\mathbb{R}^m}$, the induced algebra homomorphism $(I^r_{m,n})^h\colon \tilde{\mathbb{D}}^r_{m+n} \to \tilde{\mathbb{D}}^r_m$ is surjective. One verifies directly that its coordinate expression is

$$
\bar{x}_{k_1 \dots k_r} = x_{k_1 \dots k_r} \tag{21}
$$

with no appearance of $x_{q_1 \dots q_r}$ with at least one q_s greater than m, $q_s = 0,1,\dots,m+n$, on the right hand side.

Let $\mathbb{D}^C_m$ be the sequence of Weil algebras determined by a nonholonomic r-jet category C. Then $I^r_{m,n}$ induces a restricted and corestricted algebra homomorphism

$$
I^C_{m,n}\colon \mathbb{D}^C_{m+n} \to \mathbb{D}^C_m\,, \quad I^C_{m,n}(\mathbb{D}^C_{m+n}) = \mathbb{D}^C_m\,, \tag{22}
$$

whose coordinate expression is of the form (21).

Consider now an arbitrary sequence $\mathbb{D}^S_m$ of Weil algebras, $\mathbb{D}^r_m \subset \mathbb{D}^S_m \subset \tilde{\mathbb{D}}^r_m$, $\mathbb{D}^S_m = \mathbb{R} \times N^S_m$, and write

$$
L^S_{m,n} = (N^S_m)^n\,, \quad L^S = \bigcup_{m,n} L^S_{m,n}\,. \tag{23}
$$

Hence $L^S_{m,n} \subset \tilde{L}^r_{m,n}$.

Definition 1. The sequence $\mathbb{D}^S_m$ is called admissible, if L^S is a subcategory of $\tilde{L}^r$.

Proposition 3. *A sequence $\mathbb{D}^S_m$ is determined by a nonholonomic r-jet category C, if and only if it is admissible.*

Proof. For an admissible sequence $\mathbb{D}^S_m$, we define

$$C(M,N) = (P^r M \times P^r N)[L^S_{m,n}]\,.$$

For $X_1 \in C_x(M,N)_y$ and $X_2 \in C_y(N,Q)_z$, $X_1 = \{u, v, \xi_1\}$, $X_2 = \{v, w, \xi_2\}$, $u \in P^r_x M$, $v \in P^r_y N$, $w \in P^r_z Q$, $\xi_1 \in L^S_{m,n}$, $\xi_2 \in L^S_{n,p}$, we set

$$X_2 \circ X_1 = \{u, w, \xi_2 \circ \xi_1\}$$

with composition in L^S on the right hand side. One verifies directly that C has all required properties. □

In particular, if $\mathbb{D}^S_m$ is an admissible sequence, then $I^r_{m,n}$ maps $\mathbb{D}^S_{m+n}$ onto $\mathbb{D}^S_m$. Further, since G^r_m acts on $(N^S_m)^n$ fiberwise, every algebra $\mathbb{D}^S_m$ is G^r_m-invariant.

Thus, in order to find all nonholonomic r-jet categories, we can proceed in the following way.

(i) We determine all G^r_m-invariant Weil algebras $\mathbb{D}^r_m \subset \mathbb{D}^S_m \subset \tilde{\mathbb{D}}^r_m$ for every m.

(ii) We restrict ourselves to the sequences satisfying (22).

(iii) We analyze under what conditions (23) is a subcategory of $\tilde{L}^r$.

Example 2. In [4], we deduced that all G^2_m-invariant subalgebras of $\tilde{\mathbb{D}}^2_m$ are $\mathbb{D}^2_m$, $\bar{\mathbb{D}}^2_m$, and $\tilde{\mathbb{D}}^2_m$. The sequences satisfying (22) are $\mathbb{D}^2_m$, $\bar{\mathbb{D}}^2_m$ and $\tilde{\mathbb{D}}^2_m$, $m \in \mathbb{N}$. They determine the categories J^2, $\bar{J}^2$ and $\tilde{J}^2$.

4. Semiholonomic 3-jet categories

A nonholonomic r-jet category C is called semiholonomic, if $C(M,N) \subset \bar{J}^r(M,N)$ for all M and N. We are going to describe the semiholonomic 3-jet categories. In the course of direct evaluations, we use the coordinate formula for the composition of semiholonomic 3-jets. In the coordinates determined by (5), if $a = (a^p_i, a^p_{ij}, a^p_{ijk}) \in \bar{L}^3_{m,n}$ and $b = (b^v_p, b^v_{pq}, b^v_{pqr}) \in \bar{L}^3_{n,p}$, then $c = b \circ a = (c^v_i, c^v_{ij}, c^v_{ijk}) \in \bar{L}^3_{m,p}$ is of the form

$$\begin{aligned} &c^v_i = b^v_p a^p_i\,, \quad c^v_{ij} = b^v_{pq} a^p_i a^q_j + b^v_p a^p_{ij}\,, \\ &c^v_{ijk} = b^v_{pqr} a^p_i a^q_j a^r_k + b^v_{pq} a^p_{ik} a^q_j + b^v_{pq} a^p_i a^q_{jk} + b^v_{pq} a^p_{ij} a^q_k + b^v_p a^p_{ijk}\,. \end{aligned} \tag{24}$$

Lemma 1. *The only subalgebra $A \subset \bar{\mathbb{D}}^3_m$ satisfying $\pi^3_2(A) = \bar{\mathbb{D}}^2_m$ is $\bar{\mathbb{D}}^3_m$.*

Proof. We prove that the kernel of the induced map $\bar{N}^3_m \to \bar{N}^2_m$ is $\overset{3}{\otimes}\mathbb{R}^{m*}$. Indeed, we deduce directly by (24) that the coordinate expression of the product in $\bar{\mathbb{D}}^3_m$ of x, $y \in \bar{N}^3_m$, $z = xy$, is

$$\begin{aligned} &z_i = 0\,, \quad z_{ij} = x_i y_j + x_j y_i\,, \\ &z_{ijk} = x_{ij} y_k + x_{ik} y_j + x_i y_{jk} + x_{jk} y_i + x_j y_{ik} + x_k y_{ij}\,. \end{aligned} \tag{25}$$

Hence the tensor Z_{ijk} with $z_{ijk} = 1$ and all other coordinates equal to zero is obtained by multiplying $X_{ij} \in \bar{N}^3_m$ and $Y_k \in \bar{N}^3_m$, where the first and second order components of X_{ij} are $x_{ij} = 1$ and zero otherwise and the first and second order components of Y_k are $y_k = 1$ and zero otherwise. □

In [4] we studied the bundles

$$\bar{J}^{r,r-1}(M,N) = \{X \in \bar{J}^r(M,N), \pi^r_{r-1}(X) \in J^{r-1}(M,N)\}$$

of semiholonomic r-jets that are holonomic up to the order $r-1$. Already in [2] we deduced that for every $X \in \bar{J}^{r,r-1}_x(M,N)_y$ there exists a unique $\sigma(X) \in J^r_x(M,N)_y$ satisfying

$$\sigma(X) \circ U = X \circ U \in (T^r_1 N)_y \quad \text{for all} \quad U \in (T^r_1 M)_x\,.$$

The difference $X - \sigma(X)$ is a well defined element of $T_y N \otimes \overset{r}{\otimes} T^*_x M$. This identifies $\bar{J}^{r,r-1}(M,N)$ with the fiber product over $M \times N$

$$J^r(M,N) \times_{M\times N} TN \otimes (\overset{r}{\otimes} T^*M / S^r T^*M)\,.$$

In the case $\bar{\mathbb{D}}^{r,r-1}_m = \bar{J}^{r,r-1}_0(\mathbb{R}^m, \mathbb{R})$, we obtain

$$\bar{\mathbb{D}}^{r,r-1}_m = \mathbb{D}^r_m \times V\,, \quad V := \overset{r}{\otimes}\mathbb{R}^{m*}/S^r\mathbb{R}^{m*}\,. \tag{26}$$

The action of G^r_m on $\bar{\mathbb{D}}^{r,r-1}_m$ is

$$X \circ g = \big(\sigma(X)\circ g, l(g_1)(X - \sigma(X))\big)\,, \tag{27}$$

where $l(g_1)$ denotes the standard action of $g_1 = \pi^r_1(g) \in GL(m,\mathbb{R})$ on V. This implies easily the following assertion from [4].

Lemma 2. *The G^r_m-invariant Weil algebras $\mathbb{D}^r_m \subset A \subset \bar{\mathbb{D}}^{r,r-1}_m$ are of the form $A = \mathbb{D}^r_m \times L$, where L is a $GL(m,\mathbb{R})$-invariant linear subspace of $\overset{r}{\otimes}\mathbb{R}^{m*}$ containing $S^r\mathbb{R}^{m*}$.*

Further, using the formulae from [4], one deduces directly the following assertion.

Lemma 3. *Let $A' = \mathbb{D}^r_m \times L'$ be another such algebra. Then the G^r_m-invariant algebra homomorphisms $A \to A'$ are in bijection with the $GL(m,\mathbb{R})$-invariant linear maps $L \to L'$.*

Going back to the case $r = 3$, Lemma 1 implies that we can restrict ourselves to the bundles $\bar{J}^{3,2}(M,N)$. In [10], G. Vosmanská deduced that all natural transformations $\bar{J}^{3,2} \to \bar{J}^{3,2}$ over the identity of J^2

form a 5-parameter family Ψ. Its coordinate expression is

$$
\begin{aligned}
&\bar a^p_i = a^p_i\,, \quad \bar a^p_{ij} = a^p_{ij} \quad \text{with} \quad a^p_{ij} = a^p_{ji}\,,\\
(28)\qquad &\bar a^p_{ijk} = a^p_{ijk} + c_1(a^p_{ikj} - a^p_{ijk}) + c_2(a^p_{jik} - a^p_{ijk})\\
&\qquad + c_3(a^p_{jki} - a^p_{ijk}) + c_4(a^p_{kij} - a^p_{ijk}) + c_5(a^p_{kji} - a^p_{ijk})\,.
\end{aligned}
$$

We introduce $\bar J^{2,3}_h Y = J^1_h(J^2_h Y) \cap \bar J^3_h Y$ and $\bar J^{2,3}(M,N) = \bar J^{2,3}_h(M \times N \to M)$. In coordinates, $\bar J^{2,3}(M,N)$ is characterized by

$$
(29)\qquad a^p_{ij} = a^p_{ji}\,, \quad a^p_{ijk} = a^p_{jik}\,,
$$

so that $\bar J^{2,3}(M,N) \subset \bar J^{3,2}(M,N)$. By (24), $\bar J^{2,3}$ is a semiholonomic 3-jet category. Further, for every $\psi \in \Psi$, $(\psi \circ \bar J^{2,3})(M,N) \subset \bar J^{3,2}(M,N)$ is a fibered submanifold and (24) implies that every $\psi \circ \bar J^{2,3}$ is a semiholonomic 3-jet category.

We do not find appropriate to discuss deeper results from representation theory in this general paper. If we consider an invariant tensor of degree 3 interpreted as a linear map $\iota\colon \overset{3}{\otimes}\,\mathbb{R}^{m*} \to \overset{3}{\otimes}\mathbb{R}^{m*}$ and assume it vanishes on $S^3\mathbb{R}^{m*}$, then the kernel of ι determines an invariant subspace of $V = \overset{3}{\otimes}\mathbb{R}^{m*}/S^3\mathbb{R}^{m*}$. By the Invariant tensor theorem, [8], all invariant tensors of degree 3 form a 6-parameter family

$$
(30)\qquad d_1x_{ijk} + d_2x_{ikj} + d_3x_{jik} + d_4x_{jki} + d_5x_{kij} + d_6x_{kji}
$$

and vanishing on $S^3\mathbb{R}^{m*}$ means

$$
(31)\qquad d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0\,.
$$

Hence (30) and (31) determine a family $\mathscr{L}$ of invariant subspaces of V. Since $x_{ijk} = x_{jki}$ and $x_{kij} = x_{ijk}$ define the same subspace, $\mathscr{L}$ is linearly generated by the subspaces

(32)

$$
L_1\colon x_{ijk} = x_{jik}\,, \quad L_2\colon x_{ijk} = x_{ikj}\,, \quad L_3\colon x_{ijk} = x_{kji}\,, \quad L_4\colon x_{ijk} = x_{jki}\,.
$$

If x_{ijk} satisfy any two equations from (32), then they are symmetric in all subscripts. We shall say that the subspaces of $\mathscr{L}$ and the zero subspace are the generic invariant subspaces of V. (We do not know any non-generic one.) The corresponding semiholonomic 3-jet categories together with $\bar J^3$ and $\bar J^{3,2}$ are also called generic. Then we can formulate our main classification result as follows.

Proposition 4. *All generic semiholonomic* 3*-jet categories are* $\bar J^3$, $\bar J^{3,2}$, J^3 *and* $\psi \circ \bar J^{2,3}$ *for all* $\psi \in \Psi$.

Proof. The subspace corresponding to $\bar{J}^{2,3}$ is L_1. The restriction of the algebra homomorphisms corresponding to the natural transformations (28) to L_1 is of the form

$$\begin{aligned}\bar{x}_{ijk} &= x_{ijk} + c_1(x_{ikj} - x_{ijk}) + c_3(x_{jki} - x_{ijk}) \\ &\quad + c_4(x_{kij} - x_{ijk}) + c_5(x_{kji} - x_{ijk})\,. \end{aligned} \tag{33}$$

One evaluates directly that (33) maps L_1 into L_2 or L_3 or L_4 for the values $c_1 = \frac{1}{2}$, $c_3 = c_4 = c_5 = 0$ or $c_1 = c_3 = c_4 = 0$, $c_5 = \frac{1}{2}$ or $c_1 = c_5 = 0$, $c_3 = c_4 = \frac{1}{3}$, respectively. □

Example 3. There is an interesting problem to geometrize the semi-holonomic 3-jet categories of the form $\psi \circ \bar{J}^{2,3}$, $\psi \in \Psi$. The simpliest case is $x_{ijk} = x_{ikj}$. This corresponds to the functor $J^2_h(J^1_h Y) \cap \bar{J}^3_h Y$ restricted to the product fibered manifolds $M \times N \to M$.

References

[1] C. Ehresmann, *Oeuvres complètes et commentés*, Parties I-1 et I-2, Cahiers Topol. Géom. Diff. XXIV (Suppl. 1 et 2), 1983.

[2] I. Kolář, *The contact of spaces with connection*, J. Differential Geom. **7** (1972), 563–570.

[3] I. Kolář, *Bundle functors of the jet type*, Differential Geom. and Applications, Proceedings, Masaryk University, Brno, 1999, 231–237.

[4] I. Kolář, *A general point of view to nonholonomic jet bundles*, Cahiers Topol. Géom. Diff. Catégoriques **XLIV** (2003), 149–160.

[5] I. Kolář, *Weil Bundles as Generalized Jet Spaces*, in: Handbook of Global Analysis, Elsevier, Amsterdam (2008), 625–665.

[6] I. Kolář, *On special types of nonholonomic contact elements*, Differential Geom. Appl. **29** (2011), 135–140.

[7] I. Kolář, W. M. Mikulski, *On the fiber product preserving bundle functors*, Differential Geom. Appl. **11** (1999), 105–115.

[8] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[9] P. Libermann, *Introduction to the theory of semi-holonomic jets*, Arch. Math. (Brno) **33** (1997), 173–189.

[10] G. Vosmanská, *Natural transformations of semi-holonomic* 3*-jets*, Arch. Math. (Brno) **31** (1995), 313–318.

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