ON THE WEILIAN PROLONGATIONS OF NATURAL BUNDLES

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Abstract. We describe the Weilian prolongations of natural bundles and certain related structures.

We start with some general remarks on the role of natural bundles EM and Weil functors T^A in differential geometry in Section 1. The main aim of the present paper is to study the Weilian prolongation $T^A(EM)$ of a natural bundle from such point of view. So we begin, in Section 2, with the prolongation of an arbitrary associated bundle $P[Q]$ with respect to a bundle functor F on the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms, where m is the dimension of the base of P and $n = \dim Q$. In Section 3 we pass to the case of a natural bundle over m-manifolds. In Proposition 1 we describe the structure of natural bundle on $F(EM) \to M$. The related natural transformations are characterized too. Then some lemmas on Weil bundles are deduced in Section 4. Further we describe the Weilian prolongation $T^A(P[Q])$ of an associated bundle. The last section is devoted to $T^A(EM)$.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [2].

1. Introduction. If we look to various differential geometric structures from general point of view, we can observe that they have certain common properties depending on the category on which they are defined. Write $\mathcal{M}f$ for the category of all manifolds and smooth maps and $\mathcal{FM}$ for the category of all fibered manifolds and fiber preserving maps. In the terminology of [2], a bundle functor D on a subcategory $\mathcal{C}$ of $\mathcal{M}f$ is a covariant functor transforming every $\mathcal{C}$-object M into a fibered manifold DM over M and every $\mathcal{C}$-morphism $f\colon M \to M'$

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into an $\mathcal{FM}$-morphism $Df\colon DM \to DM'$ over f. For example, the tangent functor T is defined on the whole category $\mathcal{M}f$. On the other hand, the cotangent functor T^* is defined on the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms. Indeed, a smooth map $f\colon M \to M'$ induces the linear map $T_xf\colon T_xM \to T_{f(x)}M'$, $x \in M$, whose dual map is $(T_xf)^*\colon T^*_{f(x)}M' \to T^*_xM$. If this is an isomorphism, we can construct $T^*_xf := ((T_xf)^*)^{-1}\colon T^*_xM \to T^*_{f(x)}M'$. In [2], several general differencies between the geometry of T and T^* are pointed out.

The bundle functors on $\mathcal{M}f_m$ are the classical natural bundles over m-manifolds in the sense of A. Nijenhuis, [4], [2]. Every such functor E is of the form $EM = P^rM[Q, l]$, where P^rM is the r-th order frame bundle of M, $l\colon G^r_m \times Q \to Q$ is a left action of its structure group G^r_m on Q, $m = \dim M$, and $Ef = P^rf[Q, l]\colon P^rM[Q, l] \to P^rM'[Q, l]$ is the morphism of associated bundles induced by the principal bundle morphism $P^rf\colon P^rM \to P^rM'$ determined by local diffeomorphism $f\colon M \to M'$. The bundle functors on the category $\mathcal{FM}_{m,n}$ of fibered (m, n)-manifolds and their local isomorphisms, that can be called natural bundles over fibered (m, n)-manifolds, are analogously described in [2].

An important general result is that the product preserving bundle functors F on $\mathcal{M}f$ are in bijection with Weil algebras, [1], [2]. The simpliest Weil algebras are $\mathbb{D}^s_k = J^s_0(\mathbb{R}^k, \mathbb{R})$. An arbitrary Weil algebra A can be interpreted as a factor algebra

$$A = \mathbb{D}^s_k/\sim\,, \tag{1}$$

[1]. Functor F determines a Weil algebra $A = F\mathbb{R}$. We shall use the so-called covariant approach to Weil functor T^A, [1]. In the case of $\mathbb{D}^s_k$, $T^{\mathbb{D}^s_k} = T^s_k$ is the classical functor of (k, s)-velocities, $T^s_kM = J^s_0(\mathbb{R}^k, M)$, and we define $\mathbb{D}^s_k$-velocity of a map $\gamma\colon \mathbb{R}^k \to M$ by $j^{\mathbb{D}^s_k}\gamma = j^s_0\gamma \in T^s_kM$. Hence the tangent functor T corresponds to the algebra of dual numbers $\mathbb{D} = \mathbb{D}^1_1 = \{a + be; a, b \in \mathbb{R}, e^2 = 0\}$. In the case of arbitrary A, we introduce the A-velocity $j^A\gamma \in T^AM$ of $\gamma\colon \mathbb{R}^k \to M$ by a suitable factorization of $j^s_0\gamma$ corresponding to (1), see [1]. Then the map $T^Af\colon T^AM \to T^AM'$ induced by $f\colon M \to M'$ is of the form

$$T^Af(j^A\gamma) = j^A(f \circ \gamma)\,, \qquad \gamma\colon \mathbb{R}^k \to M\,. \tag{2}$$

We have $F = T^{F\mathbb{R}}$. An important fact is that the natural transformations of two Weil functors T^A and T^B are in bijection with the algebra homomorphisms $\mu\colon A \to B$. We write $\mu_M\colon T^AM \to T^BM$ for the corresponding map.

In [1] we collected several examples showing that the Weil algebra technique is very efficient in various concrete evaluations in differential geometry.

We remark that the Weil algebra technique can be applied to many other classes of geometric functors. The most important case are the fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphisms as base maps, [1], [3].

2. F-prolongation of associated bundles. Consider a principal bundle $P(M,G)$, $m = \dim M$. Its s-th principal prolongation W^sP is the bundle of s-jets $j^s_{(0,e)}\varphi$ of local $\mathcal{PB}$-isomorphisms $\varphi\colon \mathbb{R}^m \times G \to P$, where $0 \in \mathbb{R}^m$ and e is the unit of G. This is a principal bundle over M with structure group

$$W^s_m G = W^s_0(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_0\,,$$

both the composition in $W^s_m G$ and its right action on W^sP being defined by composition of jets, [1]. Every diffeomorphism $\varphi_0\colon \mathbb{R}^m \to \mathbb{R}^m$, $\varphi_0(0) = 0$, and every map $\varphi_1\colon \mathbb{R}^m \to G$ determine a $\mathcal{PB}$-isomorphism

$$\varphi\colon \mathbb{R}^m \times G \to \mathbb{R}^m \times G\,, \quad \varphi(x,g) = \big(\varphi_0(x), \varphi_1(x)g\big)\,, \tag{3}$$

$x \in \mathbb{R}^m$, $g \in G$. Passing to s-jets, we obtain an identification $W^s_m G = G^s_m \times T^s_m G$. As a group, $W^s_m G$ is semidirect product $G^s_m \rtimes T^s_m G$, [1, p. 150].

In general, let $P(M,G)$ and $P'(M',G)$ be two principal G-bundles and φ, $\varphi'\colon P \to P'$ be two $\mathcal{PB}$-morphisms with the underlying base maps $\underline{\varphi}$, $\underline{\varphi}'\colon M \to M'$, $\underline{\varphi}(x) = \underline{\varphi}'(x)$, $x \in M$. By equivariancy, if $j^s_u\varphi = j^s_u\varphi'$ at a point $u \in P_x$, then $j^s_{ug}\varphi = j^s_{ug}\varphi'$ for every $g \in G$. Simplifying the notation from [2], we write $j^s_x\varphi = j^s_x\varphi'$ in such a case and we say that φ and φ' have s-th order contact at $x \in M$.

Consider the category $\mathcal{PB}_m(G)$ of principal G-bundles with m-dimensional bases and $\mathcal{PB}$-morphisms with local diffeomorphisms as base maps. A gauge natural bundle is a functor $D\colon \mathcal{PB}_m(G) \to \mathcal{FM}$ such that every DP is over the same base as P and every $Df\colon DP \to DP'$ has the same base map as f, [2]. Such a functor is said to be of order s, if $j^s_x\varphi = j^s_x\varphi'$ implies

$$(D\varphi)_x = (D\varphi')_x\colon (DP)_x \to (DP')_{\underline{\varphi}(x)}\,.$$

Write $Z = D(\mathbb{R}^m \times G)_0$ and define a left action

$$\lambda_D\colon W^s_m G \times Z \to Z$$

as follows. Given $(j_0^s\varphi_0, j_0^s\varphi_1) \in G_m^s \times T_m^s G$, we construct (3) and we set

$$\lambda_D(j_0^s\varphi_0, j_0^s\varphi_1) = (D\varphi)_0 \colon Z \to Z\,. \tag{4}$$

Then Proposition 51.6 from [2] implies

Lemma 1. *DP is an associated bundle*

$$DP = W^s P[Z, \lambda_D]\,. \tag{5}$$

Consider two such functors $D_i P = W^s P[Z_i, \lambda_{D_i}]$, $i = 1, 2$. Every natural transformation $\psi\colon D_1 \to D_2$ determines a $W_m^s G$-equivariant map

$$\psi_0 := (\psi_{\mathbb{R}^m \times G})_0 \colon Z_1 \to Z_2\,.$$

Conversely, every $W_m^s G$-equivariant map $\psi\colon Z_1 \to Z_2$ defines a natural transformation $W^s P[Z_1, \lambda_1] \to W^s P[Z_2, \lambda_2]$ by $\{u, z\} \mapsto \{u, \psi(z)\}$, $u \in W^s P$, $z \in Z_1$.

Consider a left action $l\colon G \times Q \to Q$ and a bundle functor F of order s on $\mathcal{FM}_{m,n}$, $n = \dim Q$. Every $\mathcal{PB}_m(G)$-morphism $f\colon P \to P'$ induces a morphism $F_Q\colon P[Q] \to P'[Q]$ of associated bundles. Then the rule $F^Q(P) = F(P[Q])$ and $F^Q(f) = F(f_Q)$ is a gauge natural bundle of order s. Write $W_F l = \lambda_{F^Q}$. Then Lemma 1 yields

Lemma 2. *$F^Q P$ is an associated bundle*

$$F^Q P = W^s P[Z, W_F l]\,. \tag{6}$$

Hence $Z = (F(\mathbb{R} \times Q))_0$ and the action $W_F l\colon W_m^s G \times Z \to Z$ has the following form. The associated bundle morphism φ_Q induced by (3) is

$$\varphi_Q(x, a) = \big(\varphi_0(x), l(\varphi_1(x))(a)\big)\,, \quad a \in Q\,, \tag{7}$$

so that

$$W_F l(j_0^s\varphi_0, j_0^s\varphi_1) = (F^Q\varphi)_0 \colon Z \to Z\,. \tag{8}$$

Consider left actions $l_i\colon G \times Q_i \to Q_i$, $i = 1, 2$. According to [2], every G-map $h\colon Q_1 \to Q_2$ defines a natural transformation $h^F\colon F^{Q_1} \to F^{Q_2}$ determined by a $W_m^s G$-map

$$h_0^F\colon F_0(\mathbb{R}^m \times Q_1) \to F_0(\mathbb{R}^m \times Q_2)\,. \tag{9}$$

Further, given F_1, F_2, every natural transformation $\psi\colon F_1 \to F_2$ defines a natural transformation $\psi^Q\colon F_1^Q \to F_2^Q$ determined by a $W_m^s G$-map

$$\psi_0^Q\colon \big(F_1(\mathbb{R}^m \times Q)\big)_0 \to \big(F_2(\mathbb{R}^m \times Q)\big)_0\,. \tag{10}$$

3. ***F*-prolongation of natural bundles.** In the case of an r-th order natural bundle E, we have $EM = P^r M[Q, l]$, where $l\colon G^r_m \times Q \to Q$. Hence we obtain, by Lemma 2,

$$F(EM) = W^s P^r M[Z, W_F l]\,. \tag{11}$$

There is a canonical injection $P^{r+s}M \hookrightarrow W^s P^r M$, $j^{r+s}_0\varphi \mapsto j^s_{(0,e)} P^r\varphi$ with a group injection $i\colon G^{r+s}_m \hookrightarrow W^s_m G^r_m$, [2]. Write $W_F l \circ (i \times \mathrm{id}_Z) =: l_F\colon G^{r+s}_m \times Z \to Z$. Then (11) implies

Proposition 1. *$F(EM)$ is an associated bundle*

$$F(EM) = P^{r+s}M[Z, l_F]\,. \tag{12}$$

The natural transformations of types (9) and (10) have the same form as in Section 2.

Remark. We can also replace E by a bundle functor defined on $\mathcal{FM}_{p,q}$, $p + q = m$. Then $P^r M$ should be replaced by the bundle of fibered frames on a fibered manifold Y with p-dimensional base and q-dimensional fibers, [1].

4. Some properties of Weil bundles. A Weil algebra $A = \mathbb{R} \times N$ is said to be of order s, if $N^{s+1} = 0$ with minimal s. (In [5], A. Weil uses the term "depth".) Then $T^A_x f$, $f\colon M \to M'$, depends on $j^s_x f$ only. This defines a system of maps

$$\begin{aligned} &\tau^A\colon J^s(M, M') \times_M T^A M \to T^A M'\,,\\ &\tau^A(j^s_x f, j^A\gamma) = j^A(f \circ \gamma)\,, \end{aligned} \tag{13}$$

$\gamma\colon \mathbb{R}^k \to M$, $\gamma(0) = x$. Clearly, $\tau^A(X_2 \circ X_1) = \tau^A(X_2) \circ \tau^A(X_1)$ with composition of s-jets on the left-hand side.

Since T^A preserves products, $T^A\mathbb{R}^m = A^m$ is the product bundle $\mathbb{R}^m \times N^m$. We have $G^s_m = \mathrm{inv}\, J^s_0(\mathbb{R}^m, \mathbb{R}^m)_0$ and the restriction of τ^A defines a left action

$$\begin{aligned} &\tau^A_m\colon G^s_m \times N^m \to N^m\,,\\ &\tau^A_m\big(j^s_0 f, (j^A\gamma_1, \dots, j^A\gamma_m)\big) = j^A f(\gamma_1, \dots, \gamma_m)\,. \end{aligned} \tag{14}$$

By construction, $T^A M$ is the associated bundle

$$T^A M = P^s M[N^m, \tau^A_m]\,. \tag{15}$$

We need the following algebraic assertion.

Lemma 3. *If A is of order s, then $N^m = \mathrm{Hom}\,(\mathbb{D}^s_m, A)$.*

Proof. Let $x_1, \dots, x_m$ be the standard generators of $\mathbb{D}^s_m$. Every algebra homomorphism $H\colon \mathbb{D}^s_m \to A$ is determined by the values $H(x_i) \in N$. Since A is of order s, these values can be prescribed arbitrarily. $\square$

By Section 1, the natural transformation $H_Q\colon T^s_m Q \to T^A Q$ determined by H over a manifold Q is of the form

$$H_Q(j^s_0 f) = j^A f(\gamma_1, \dots, \gamma_m)\,, \quad H(x_i) = j^A \gamma_i,\ f\colon \mathbb{R}^m \to Q\,. \tag{16}$$

5. Weilian prolongations of associated bundles. First we describe $T^A(P[Q])$. Since T^A preserves products, we have $T^A(\mathbb{R}^m \times Q) = T^A\mathbb{R}^m \times T^A Q$. Hence $Z = N^m \times T^A Q$. We define a map $W_A l\colon W^s_m G \times (N^m \times T^A Q) \to N^m \times T^A Q$ by

$$W_A l\big((X,Y),(H,K)\big) = \big(\tau^A_m(X)(H), T^A l(H_G(Y), K)\big)\,, \tag{17}$$

$X \in G^s_m$, $Y \in T^s_m G$, $H \in N^m$, $K \in T^A Q$ and in H_G we interpret H as an algebra homomorphism $\mathbb{D}^s_m \to A$. By [1], $H_G\colon T^s_m G \to T^A G$ is a group homomorphism.

This is an instructive exercise to verify formally that (17) is a left action, but it is a consequence of the following assertion.

Proposition 2. *We have $W_A l = W_{T^A} l$. Hence*

$$T^A(P[Q]) = W^s P[N^m \times T^A Q, W_A l]\,. \tag{18}$$

Proof. We evaluate (8) in the case $F = T^A$. Consider $j^A\gamma \in N^m$ and $j^A\delta \in T^A Q$. According to (7), $W_{T^A} l(j^s_0\varphi_0, j^s_0\varphi_1)(j^A\gamma, j^A\delta) = T^A\varphi_Q(j^A\gamma, j^A\delta) = \big(j^A(\varphi_0 \circ \gamma), T^A l\big(j^A(\varphi_1 \circ \gamma), j^A\delta\big)\big)$. By (16), $j^A(\varphi_1 \circ \gamma) = H_G(j^s_0\varphi_1)$. □

We also describe the natural transformations from Section 2 in the Weilian situation. In (9) with $F = T^A$, we have $T^A_0(\mathbb{R}^m \times Q_i) = \mathbb{R}^m \times N^m \times T^A Q_i$, $i = 1, 2$ and $T^A(\mathrm{id}_{\mathbb{R}^m} \times h) = \mathrm{id}_{T^A\mathbb{R}^m} \times T^A h$. Hence $h^{T^A} = \mathrm{id}_{N^m} \times T^A h$.

In (10) with $F_1 = T^A$, $F_2 = T^B$ and ψ determined by an algebra homomorphism $\mu\colon A \to B$, we have $\psi^Q_0\colon N^m_A \times T^A Q \to N^m_B \times T^B Q$ of the form $\big((\mu \mid N_A)^m \times \mu_Q\big)$, provided N_A or N_B denotes the nilpotent part of A or B, respectively.

6. The case of $T^A(EM)$. Now we can proceed analogously to Section 3. In the case of a natural bundle $EM = P^r M[Q, l]$, we first obtain $T^A(EM) = W^s P^r M[Z, W_A l]$, $Z = N^m \times T^A Q$. Using the injection $i\colon G^{r+s}_m \hookrightarrow W^s_m G^r_m$, we define $l_A = W_A l \circ (i \times \mathrm{id}_Z)\colon G^{r+s}_m \times Z \to Z$. Then Proposition 2 implies

Proposition 3. *$T^A(EM)$ is an associated bundle*

$$T^A(EM) = P^{r+s} M[N^m \times T^A Q, l_A]\,. \tag{19}$$

The two types of natural transformations studied in Section 5 are expressed by the same formulae even in the situation of Proposition 3.

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