SOME PROPERTIES OF FIBER PRODUCT PRESERVING BUNDLE FUNCTORS

Ivan Kolář

Abstract. Let F be a fiber product preserving bundle functor on the category $\mathcal{FM}_m$ of the proper base order r. We deduce that the r-th principal gauge prolongation W^rP of a principal bundle P is a reduction of the principal bundle W^FP constructed in [1]. In certain special cases, we solve the problem up to what extent F is determined by the underlying group homomorphism.

According to [7], every fiber product preserving bundle functor F on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered morphisms with local diffeomorphisms as base maps can be characterized by a triple (A, H, t), where A is a Weil algebra, $H\colon G^r_m \to \mathrm{Aut}\, A$ is a group homomorphism of the r-th jet group in dimension m into the group of all algebra automorphisms of A and $t\colon \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism, $\mathbb{D}^r_m = J^r_0(\mathbb{R}^m, \mathbb{R})$. In [1], the principal bundle W^FP was defined for every principal bundle P and it was deduced that the F-prolongation of an associated bundle $E = P[S]$ is an associated bundle $FE = W^FP[T^AS]$.

In Definition 1 of the present paper, we introduce the concept of the proper base order of F. Lemma 1 is a general algebraic result concerning equivariant algebra homomorphisms $\mathbb{D}^r_m \to A$. Using Lemma 1, we deduce, in Proposition 1, that if r is the proper base order of F, then the canonical map $J^rY \to FY$ is injective for every fibered manifold Y. This implies easily that the r-th principal gauge prolongation W^rP of P, [2], [6], is a reduction of W^FP. Hence FE can be interpreted as a fiber bundle associated to W^rP as well.

The second part of the present paper is devoted to the following problem. Given $H\colon G^r_m \to \mathrm{Aut}\, A$, what are all possible $t\colon \mathbb{D}^r_m \to A$ such that (A, H, t) is a fiber product preserving bundle functor of $\mathcal{FM}_m$? In Section 4 we deduce that for holonomic and semiholonomic r-jets,

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$r \geq 2$, the only two possibilities are the r-th horizontal and the r-th vertical prolongation of a fibered manifold. In the case of nonholonomic r-jets, there are more possibilities. In Section 5 we describe completely the case $r = 2$.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notations from the book [6].

1. Decomposition lemma. By [4], the group Aut $\mathbb{D}^r_m$ of all algebra automorphisms of $\mathbb{D}^r_m$ coincides with G^r_m. The corresponding map $G^r_m \times \mathbb{D}^r_m \to \mathbb{D}^r_m$ is the jet composition $(g, X) \mapsto X \circ g$, $g \in G^r_m$, $X \in \mathbb{D}^r_m$. Consider an arbitrary Weil algebra A and a group homomorphism $H \colon G^r_m \to \text{Aut } A$. An algebra homomorphism $\mu \colon \mathbb{D}^r_m \to A$ is called H-equivariant, if

$$\mu(X \circ g) = H(g)\big(\mu(X)\big)\,, \qquad X \in \mathbb{D}^r_m,\ g \in G^r_m\,.$$

In particular, the jet projection $\pi^r_s \colon G^r_m \to G^s_m$, $s \leq r$, is a group homomorphism and the jet projection $\beta^r_s \colon \mathbb{D}^r_m \to \mathbb{D}^s_m$ is a π^r_s-equivariant algebra homomorphism.

Lemma 1. *For every H-equivariant algebra homomorphism $\mu \colon \mathbb{D}^r_m \to A$, there exists $s \leq r$ and an injective H_s-equivariant algebra homomorphism $i \colon \mathbb{D}^s_m \to A$ such that $\mu = i \circ \beta^r_s$, where $H_s \colon G^s_m \to \text{Aut } A$ is the group homomorphism determined by $H = H_s \circ \pi^r_s$.*

Proof. Write $I = \text{Ker } \mu$ and $L^r_{m,m} = J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$. Then $G^r_m \subset L^r_{m,m}$ is open and dense and the jet composition $\mathbb{D}^r_m \times L^r_{m,m} \to \mathbb{D}^r_m$ is a polynomial map. We know that $X \in I$ and $g \in G^r_m$ implies $X \circ g \in I$. By continuity, this holds for every $g \in L^r_{m,m}$.

Now we can apply our result from [3]. Write $\widetilde{I}$ for the inverse image of I in the algebra $\mathcal{E}(m)$ of germs of smooth functions on $\mathbb{R}^m$ at 0. Then $\widetilde{I}$ has the substitution property: if $\xi \in \widetilde{I}$ and γ is a germ of an origin preserving smooth map $\mathbb{R}^m \to \mathbb{R}^m$, then $\xi \circ \gamma \in \widetilde{I}$. By Lemma 2 from [3], $\widetilde{I} = \mathfrak{m}^{s+1}(m)$, where $\mathfrak{m}(m)$ is the maximal ideal of $\mathcal{E}(m)$. Hence $I = \mathfrak{m}^{s+1}(m)\big/\mathfrak{m}^{r+1}(m) =: I^r_s$. Write $K^r_s = \text{Ker } \pi^r_s$. Then we have $i \colon \mathbb{D}^s_m \to A$, $i(X + I^r_s) = \mu(X)$ and $H_s(gK^r_s)\big(i(X + I^r_s)\big) = i(X \circ g + I^r_s)$. □

2. Fiber product preserving bundle functors. We reformulate the description of a fiber product preserving bundle (in short: f.p.p.b.) functor F on $\mathcal{FM}_m$, [7], by using an additional concept of the proper base order of F. By Section 13 of [7], F has finite order. The construction of product bundles and product morphisms defines an injection

$i\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, where $\mathcal{M}f$ denotes the category of all manifolds and $\mathcal{M}f_m$ means the category of m-dimensional manifolds and local diffeomorphisms. In general, a bundle functor Φ on $\mathcal{M}f_m \times \mathcal{M}f$ is said to be of order r in the first factor, if for every g, $g'\colon M \to M'$ and $f\colon N \to N'$, $j^r_x g = j^r_x g'$ implies $\Phi_x(g,f) = \Phi_x(g',f)\colon \Phi_x(M,N) \to \Phi_{g(x)}(M',N')$, $x \in M$, [7]. Consider the case $\bar{F} := F \circ i$.

Definition 1. The minimal order of $\bar{F}$ in the first factor is called the proper base order of F.

We recall that the product preserving bundle functors on $\mathcal{M}f$ are in bijection with the functors T^A determined by a Weil algebra A and the natural transformations $T^{A_1} \to T^{A_2}$ are in bijection with the algebra homomorphisms $\mu\colon A_1 \to A_2$, [4], [6]. We write $\mu_N\colon T^{A_1}N \to T^{A_2}N$ for the value of μ on N.

According to [7], F induces a functor F^0 on $\mathcal{M}f$ by $F^0 N = \bar{F}_0(\mathbb{R}^m, N)$ and $F^0 f\colon F^0 N_1 \to F^0 N_2$ is the restriction and corestriction of $\bar{F}(\mathrm{id}_{\mathbb{R}^m}, f)$, $f\colon N_1 \to N_2$. Since F^0 preserves products, it is a Weil functor T^A. Consider further an origin preserving diffeomorphism φ of $\mathbb{R}^m$. If r is the proper base order of F, then the restriction and corestriction of $\bar{F}(\varphi, \mathrm{id}_N)$ over $0 \in \mathbb{R}^m$ depends on $g = j^r_0\varphi \in G^r_m$ only. The rule $g \mapsto \bar{F}_0(\varphi, \mathrm{id}_N) =: H(g)_N\colon T^A N \to T^A N$ defines a group homomorphism $H\colon G^r_m \to \mathrm{Aut}\, A$. So we have an action H_N of G^r_m on $T^A N$ and each $T^A f\colon T^A N_1 \to T^A N_2$ is an H-equivariant map. Then $\bar{F}(M,N)$ coincides with the associated bundle $P^r M[T^A N, H_N]$ and $\bar{F}(g,f)$ is the corresponding morphism $P^r g[T^A f]$ of associated bundles, see e.g. [6].

Conversely, A and $H\colon G^r_m \to \mathrm{Aut}\, A$ define a bundle functor (A,H) on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$(1) \quad (A,H)(M,N) = P^r M[T^A N, H_N]\,, \quad (A,H)(g,f) = P^r g[T^A f]\,.$$

Clearly, r is the minimal order of (A,H) in the first factor, iff H cannot be factorized through a group homomorphism $G^{r-1}_m \to \mathrm{Aut}\, A$.

Further, every section σ of a fibered manifold $p\colon Y \to M$ can be interpreted as a base preserving morphism $\bar{\sigma}\colon M \times pt \to Y$, where pt denotes a singleton. Then $F\bar{\sigma}$ is identified with a section of $FY \to M$.

Lemma 2. *If r is the proper base order of F, then $(F\bar{\sigma})(x)$ depends on $j^r_x\sigma$ only, $x \in M$.*

Proof. By locality, we may assume $Y = M \times N$ and σ is a "constant" section $x \mapsto (x,c)$, $c \in N$. Then (1) implies our claim. □

Thus we obtain a natural transformation

$$\widetilde{t}_Y\colon J^r Y \to FY\,, \quad \widetilde{t}_Y(j^r_x\sigma) = (F\bar{\sigma})(x)\,.$$

A simple analysis shows that $\widetilde{t}$ is equivalent to an equivariant algebra homomorphism $t\colon \mathbb{D}^r_m \to A$, [7]. In the product case $Y = M \times N$, we have $t_N\colon T^r_m N \to T^A N$ and

$$\widetilde{t}_{M\times N}\colon P^r M[T^r_m N] \to P^r M[T^A N]\,, \quad \widetilde{t}_{M\times N}\big(\{u, X\}\big) = \{u, t_N(X)\}\,,$$

$u \in P^r M$, $X \in T^r_m N$.

Conversely, every such triple (A, H, t) defines a f.p.p.b. functor F on $\mathcal{FM}_m$ by $F(M \times N \to M) = (A, H)(M, N) = P^r M[T^A N]$ and $FY \subset F(M \times Y \to M) = P^r M[T^A Y]$ is the subset characterized by

$$t_N(u) = T^A p(X)\,, \quad \{u, X\} \in P^r M[T^A Y]\,, \tag{2}$$

where we use $P^r M \subset T^r_m M$. For an $\mathcal{FM}_m$-morphism $f\colon Y_1 \to Y_2$ with base map $\underline{f}$, $Ff\colon FY_1 \to FY_2$ is the restriction and corestriction of $P^r \underline{f}[T^A f]$. Clearly, r is the proper base order of F, iff $t\colon \mathbb{D}^r_m \to A$ cannot be factorized through an algebra homomorphism $\mathbb{D}^{r-1}_m \to A$.

Proposition 1. *If r is the proper base order of F, then $\widetilde{t}_Y\colon J^r Y \to FY$ are injective maps.*

Proof. This follows directly from Lemma 1. Since r is the proper base order, we have $r = s$. □

The simpliest examples of f.p.p.b. functors are the vertical Weil bundle

$$V^A Y = \bigcup_{x\in M} T^A(Y_x)\,,$$

the r-th jet prolongation $J^r Y =: J^r_h Y$, that we call the r-th horizontal prolongation in this general context, the r-th vertical jet prolongation

$$J^r_v Y = \bigcup_{x\in M} J^r_x(M, Y_x)$$

and their iterations. In general, if F_1 and F_2 are two f.p.p.b. functors on $\mathcal{FM}_m$, then the functor $F_3 Y = F_1 Y \times_M F_2 Y$, $F_3 f = F_1 f \times_{\underline{f}} F_2 f$ preserves fiber products as well. If $F_i = (A_i, H_i, t_i)$, $i = 1, 2, 3$, then A_3 is the sum $A_1 \oplus A_2$, i.e. the subset of $A_1 \times A_2$ formed by all pairs with equal real parts, [4]. Further we have $\mathrm{Aut}\, A_1 \times \mathrm{Aut}\, A_2 \subset \mathrm{Aut}\,(A_1 \oplus A_2)$, $r_3 = \max(r_1, r_2)$, $H_3(g) = \big(H_1(\pi^{r_3}_{r_1} g), H_2(\pi^{r_3}_{r_2} g)\big)$, $g \in G^{r_3}_m$ and $t_3(X) = \big(t_1(\beta^{r_3}_{r_1} X), t_2(\beta^{r_3}_{r_2} X)\big)$, $X \in \mathbb{D}^{r_3}_m$. We write $F_3 = F_1 \oplus F_2$ and we say F_3 is the fiber product of F_1 and F_2.

The natural transformations $(A_1, H_1, t_1) \to (A_2, H_2, t_2)$ are in bijection with equivariant algebra homomorphisms $\mu\colon A_1 \to A_2$ satisfying $t_2 = \mu \circ t_1$, [7]. We write $\widetilde{\mu}_Y\colon F_1 Y \to F_2 Y$ and we have

$$\widetilde{\mu}_Y\big(\{u, X\}\big) = \big\{u, \mu_Y(X)\big\}\,, \quad u \in P^r M,\ X \in T^{A_1} Y\,,$$

where $\mu_Y\colon T^{A_1}Y \to T^{A_2}Y$ is the manifold version of μ. In the product case $Y = M \times N$, we have

$$\widetilde{\mu}_{M\times N}\big(\{u, X\}\big) = \big\{u, \mu_N(X)\big\}\,, \quad u \in P^rM,\ X \in T^{A_1}N\,.$$

3. F-prolongation of associated bundles. The r-th principal gauge prolongation of a principal bundle $P(M,G)$ is the bundle W^rP of r-jets of local principal bundle isomorphisms

$$j^r_{(0,e)}\psi\,, \quad \psi\colon \mathbb{R}^m \times G \to P\,, \quad 0 \in \mathbb{R}^m\,,\ e \in G\,.$$

This is a principal bundle over M with structure group $W^r_mG = W^r_0(\mathbb{R}^m \times G)$, [6]. We have $W^rP = P^rM \times_M J^rP$ and $W^r_mG = G^r_m \rtimes T^r_mG$ with the group composition

$$(g_1, X_1)\,(g_2, X_2) = \big(g_1 \circ g_2, T^r_m\gamma(X_1 \circ g_2, X_2)\big)\,,$$

where $\gamma\colon G \times G \to G$ is the group composition of G. For an arbitrary fibered manifold $p\colon Y \to M$, one defines $W^rY = P^rM \times_M J^rY$.

In [1], the authors introduced formally $W^FY = P^rM \times_M FY$. For every natural transformation $\widetilde{\mu}_Y\colon F_1Y \to F_2Y$, they defined

$$W^\mu_Y\colon W^{F_1}Y \to W^{F_2}Y\,, \quad \{u, X\} \mapsto \big\{u, \widetilde{\mu}_Y(X)\big\}\,, \quad u \in P^rM\,,\ X \in F_1Y\,.$$

In particular, $W^t_Y\colon W^rY \to W^FY$. Hence Proposition 1 implies

Corollary. *If r is the proper base order of F, then $W^t_Y\colon W^rY \to W^FY$ are injective maps.*

For a principal bundle $P(M,G)$, W^FP is a principal bundle over M, the structure group of which is $W^A_HG = G^r_m \rtimes_H T^AG$ with the group composition

$$(g_1, X_1)\,(g_2, X_2) = \big(g_1 \circ g_2, T^A\gamma\big(H_G(g_2^{-1})(X_1), X_2\big)\big)\,,$$

[1]. The map $\mathrm{id}_{G^r_m} \times t_G\colon W^r_mG = G^r_m \times T^r_mG \to G^r_m \times T^AG = W^A_HG$ is a group homomorphism. Then Proposition 1 implies easily

Proposition 2. *If r is the proper base order of F, then $W^t_P\colon W^rP \hookrightarrow W^FP$ is a reduction to the subgroup $W^r_mG \hookrightarrow W^A_HG$.*

According to [1], a left action $l\colon G \times S \to S$ induces a left action $W^A_Hl\colon W^A_HG \times T^AS \to T^AS$,

$$W^A_Hl\big((g, X), Z\big) = H(g)_S\big(T^Al(X, Z)\big)\,, \quad g \in G^r_m,\ X \in T^AG,\ Z \in T^AS\,.$$

Consider the associated bundle $E = P[S, l]$. Its frame map $(u, \{u, y\}) \mapsto y$, $u \in P$, $y \in S$ can be interpreted as a base preserving morphism

$$\varphi\colon P \times_M E \to M \times S\,.$$

Applying F, we obtain

$$F\varphi\colon FP \times_M FE \to P^r M[T^A S]\,.$$

Then FE is an associated bundle $W^F P[T^A S, W^A_H l]$, the frame map of which $\varphi_F\colon W^F P \times_M FE \to T^A S$ is of the form

$$\varphi_F\big((u,X),Z\big) = \widetilde{u}^{-1}\big(F\varphi(X,Z)\big)\,, \quad u \in P^r M, X \in FP, Z \in FE\,,$$

where $\widetilde{u}$ is the frame map of $P^r M[T^A S]$ corresponding to $u \in P^r M$, [1].

Adding the group homomorphism $\operatorname{id}_{G^r_m} \times t_G\colon W^r_m G \to W^A_H G$, we obtain a left action $W^r_H l\colon W^r_m G \times T^A S \to T^A S$,

$$W^r_H l\big((g,X),Z\big) = H_S(g) T^A l\big(t_G(X),Z\big)\,.$$

Then Proposition 2 implies directly

Proposition 3. *Let $E = P[S,l]$ be an associated bundle and r be the proper base order of F. Using $W^t_P\colon W^r P \hookrightarrow W^F P$, we can interpret FE as an associated bundle $W^r P[T^A S, W^r_H l]$ with the frame map $\varphi^r_F\colon W^r P \times_M FE \to T^A S$,*

$$\varphi^r_F\big(\big(u, \widetilde{t}_P(X)\big), Z\big) = \widetilde{u}^{-1}\big(F\varphi\big(\widetilde{t}_P(X), Z\big)\big)\,.$$

One verifies easily that the frame map $\widetilde{j^r_{(0,e)}\psi}\colon F_x E \to T^A S$ can be constructed in the following geometric way. We have $F(\mathbb{R}^m \times S) = P^r\mathbb{R}^m[T^A S] = \mathbb{R}^m \times T^A S$, where we use the identification $P^r\mathbb{R}^m = \mathbb{R}^m \times G^r_m$ defined by the translations on $\mathbb{R}^m$. The local principal bundle isomorphism $\psi\colon \mathbb{R}^m \times G \to P$ induces a local isomorphism of associated bundles $\psi_S\colon \mathbb{R}^m \times S \to E$. Applying F, we obtain $F\psi_S\colon \mathbb{R}^m \times T^A S \to FE$. This map is restricted and corestricted into a diffeomorphism $\{x\} \times T^A S \to F_x E$, that is the inverse map to $\widetilde{j^r_{(0,e)}\psi}$.

4. Algebra homomorphisms compatible with H. The bundles of nonholonomic r-jets $\widetilde{J}^r(M,N) \supset J^r(M,N)$ represent an important generalization of the classical bundles of holonomic r-jets, [4]. C. Ehresmann defined the composition of nonholonomic r-jets

$$Z \circ X \in \widetilde{J}^r_x(M,Q)_z \quad \text{for every} \quad X \in \widetilde{J}^r_x(M,N)_y \quad \text{and} \quad Z \in \widetilde{J}^r_y(N,Q)_z\,,$$

that coincides with the classical jet composition in the holonomic case, see e.g. [4]. In [5], we introduced the general concept of nonholonomic r-jet category C as a subcategory

$$J^r(M,N) \subset C(M,N) \subset \widetilde{J}^r(M,N)\,.$$

For every m, we define a Lie group $G^C_m = \operatorname{inv} C_0(\mathbb{R}^m,\mathbb{R}^m)_0$, [5], and a Weil algebra $\mathbb{D}^C_m = C_0(\mathbb{R}^m,\mathbb{R}) = \mathbb{R} \times N^C_m$, $N^C_m = C_0(\mathbb{R}^m,\mathbb{R})_0$. We have the canonical injections $i^C_m\colon G^r_m \to G^C_m$ and $\iota^C_m\colon \mathbb{D}^r_m \to \mathbb{D}^C_m$ in both group and algebra cases. The jet composition determines an injection $G^C_m \hookrightarrow \operatorname{Aut} \mathbb{D}^C_m$.

Definition 2. For a fibered manifold $p\colon Y \to M$, we define its horizontal C-prolongation

$$C_hY = \left\{X \in C(M,Y), (j^r_{\beta X}p) \circ X = j^r_{\alpha X} \operatorname{id}_M \right\}$$

and its vertical C-prolongation

$$C_vY = \bigcup_{x\in M} C_x(M, Y_x) \subset C(M,Y)\,.$$

Both C_h and C_v are f.p.p.b. functors on $\mathcal{FM}_m$ with $A = \mathbb{D}^C_m$. In both cases, the underlying group homomorphism is $i^C_m\colon G^r_m \to G^C_m$. In the first case, the algebra homomorphism is $\iota^C_m\colon \mathbb{D}^r_m \to \mathbb{D}^C_m$, while in the second one it is the zero homomorphism $\mathbb{D}^r_m \to \mathbb{D}^C_m$, $(x,n) \mapsto (x,0)$, $x \in \mathbb{R}$, $n \in N^r_m$.

This leads us to the following GENERAL PROBLEM. Let $H\colon G^r_m \to \operatorname{Aut} A$ be a group homomorphism. What are all possible algebra homomorphisms $t\colon \mathbb{D}^r_m \to A$ such that (A,H,t) is a f.p.p.b. functor on $\mathcal{FM}_m$? IN OTHER WORDS: up to what extent a f.p.p.b. functor F on $\mathcal{FM}_m$ is determined by its restriction $\bar F = F \circ i$ to $\mathcal{M}f_m \times \mathcal{M}f$?

First of all, we discuss the case $A = \mathbb{D}^r_m$ and $H = \operatorname{id}_{G^r_m} =: i^r_m$.

Lemma 3. *For $r \geq 2$, the only two i^r_m-equivariant algebra homomorphisms $\mathbb{D}^r_m \to \mathbb{D}^r_m$ are $\operatorname{id}_{\mathbb{D}^r_m}$ and the zero homomorphism. For $r = 1$, all possibilities are $(x,n) \mapsto (x,kn)$, $k \in \mathbb{R}$.*

Proof. Consider first the case $r = 2$. Write x_i, x_{ij} for the canonical coordinates on N^2_m. By standard representation results, [6], a G^1_m-equivariant linear map $f\colon N^2_m \to N^2_m$ is of the form

$$\bar x_i = k_1 x_i\,, \quad \bar x_{ij} = k_2 x_{ij}\,.$$

The equivariancy of f with respect to $(\delta^i_j, a^i_{jl}) \in G^2_m$ means

$$k_2(x_{ij} + a^l_{ij}x_l) = k_2 x_{ij} + a^l_{ij} k_1 x_l\,.$$

This implies $k_1 = k_2 =: k$. The homomorphism condition yields $(kx_i)(kx_j) = k(x_ix_j)$. Hence $k^2 = k$, i.e. $k = 0, 1$. Further, for $r = 1$ the homomorphism condition is automatically satisfied, because of $nn' = 0$ for all $n, n' \in N^1_m$. This yields our second claim. For $r > 2$, a standard recurrence procedure leads to the first claim. □

In Section 2 we introduced the functors J^r_h and J^r_v. If $r = 1$, we further take into account that $J^1_x(M,N)_y = T_yN \otimes T^*_xM$ is a vector space. Hence we can define

$$J^{1,k}Y = \{X \in J^1(M,Y), Tp(X) = k\,\mathrm{id}_{T_\alpha X M}\}$$

for every $k \in \mathbb{R}$. Clearly, $J^{1,0}Y = J^1_vY$ and $J^{1,1}Y = J^1_hY$. Then Lemma 3 yields directly

Proposition 4. *For $r \geq 2$, the only two f.p.p.b. functors with the underlying group homomorphism i^r_m are J^r_h and J^r_v. If $r = 1$, all of them form a one-parameter family $J^{1,k}$, $k \in \mathbb{R}$.*

An important subcategory of $\widetilde{J}^r$ is the category $\bar{J}^r$ of semiholonomic r-jets, [4]. In general, a nonholonomic r-jet category C will the called semiholonomic, if $C(M,N) \subset \bar{J}^r(M,N)$ for every M and N. If C is semiholonomic, then we find easily from the proof of Lemma 3 that the values of every i^C_m-equivariant algebra homomorphism $\mathbb{D}^r_m \to \mathbb{D}^C_m$ lie in $\mathbb{D}^r_m \subset \mathbb{D}^C_m$. Then Lemma 3 implies that for $r \geq 2$ the only two possibilities are i^C_m and the zero homomorphism. Thus we have proved

Proposition 5. *If C is a semiholonomic r-jet category, $r \geq 2$, then the only two f.p.p.b. functors on $\mathcal{FM}_m$ with underlying group homomorphism $i^C_m\colon G^r_m \to G^C_m \subset \mathrm{Aut}\,\mathbb{D}^C_m$ are C_h and C_v.*

We remark that the construction of fiber products of f.p.p.b. functors clarifies that we can have more possibilities in some further cases. For example, consider the algebra $\mathbb{D}^r_m \oplus \mathbb{D}^r_m$ and the injection $\delta^r_m\colon G^r_m \to \mathrm{Aut}\,(\mathbb{D}^r_m \oplus \mathbb{D}^r_m)$ from Section 2. Then $J^r_h \oplus J^r_h$, $J^r_h \oplus J^r_v$ and $J^r_v \oplus J^r_v$ are f.p.p.b. functors on $\mathcal{FM}_m$ with the underlying group homomorphism δ^r_m.

5. The nonholonomic case. To outline what can happen in the case of the category $C = \widetilde{J}^r$ of all nonholonomic r-jets, we discuss the order $r = 2$ in detail. We write $\widetilde{G}^2_m = \mathrm{inv}\,\widetilde{J}^2_0(\mathbb{R}^m,\mathbb{R}^m)_0$, $\widetilde{\mathbb{D}}^2_m = \widetilde{J}^2_0(\mathbb{R}^m,\mathbb{R})$ and $\widetilde{i}^2_m\colon G^2_m \to \widetilde{G}^2_m$, $\widetilde{\iota}^2_m\colon \mathbb{D}^2_m \to \widetilde{\mathbb{D}}^2_m$ for the canonical injections. We have $N^2_m = (x_i, x_{ij})$, $\widetilde{N}^2_m = (y_i, y_{0i}, y_{ij})$.

Lemma 4. *All $\widetilde{i}^2_m$-equivariant algebra homomorphisms $\mathbb{D}^2_m \to \widetilde{\mathbb{D}}^2_m$ form two one-parameter families I and II of the common form*

$$y_i = kx_i\,, \quad y_{0i} = lx_i\,, \quad y_{ij} = klx_{ij} \tag{3}$$

with $I\colon k \in \mathbb{R}$, $l = 1$ and $II\colon k = 0$, $l \in \mathbb{R}$.

Proof. By the classical representation theory, a G^1_m-equivariant linear map $N^2_m \to \widetilde{N}^2_m$ is of the form

$$y_i = kx_i\,, \quad y_{0i} = lx_i\,, \quad y_{ij} = hx_{ij}\,.$$

The equivariancy with respect to $(\delta^i_j, a^i_{jl}) \in G^2_m$ implies $h = k$ as in the proof of Lemma 3. The homomorphism condition yields $k = kl$. $\square$

We geometrize this result by using the functor $J^{1,k}$. We recall $J^{1,0} = J^1_v$ and $J^{1,1} = J^1_h$.

Proposition 6. *All f.p.p.b. functors on $\mathcal{FM}_m$ with the underlying group homomorphism $\widetilde{i}^2_m\colon G^2_m \to \widetilde{G}^2_m \subset \mathrm{Aut}\,\widetilde{\mathbb{D}}^2_m$ form two one-parameter families $I \equiv J^{1,k} \circ J^1_h$ and $II \equiv J^1_v \circ J^{1,l}$.*

Proof. We apply the general description of the iteration of two f.p.p.b. functors on $\mathcal{FM}_m$ from [1]. If $F = (A, H, t)$ and $E = (B, K, u)$, $K\colon G^s_m \to \mathrm{Aut}\,B$, $u\colon \mathbb{D}^s_m \to B$ are two such functors, then the Weil algebra of $F \circ E$ is $A \otimes B$ and the algebra homomorphism $v\colon \mathbb{D}^{r+s}_m \to \mathrm{Aut}\,(A \otimes B)$ is of the form

$$v = t_B \circ T^r_m u \circ \iota^{r,s}_m\,,$$

where $t_B\colon T^r_m B \to T^A B = A \otimes B$ is the value of t on B, $T^r_m u\colon T^r_m \mathbb{D}^s_m = \mathbb{D}^r_m \otimes \mathbb{D}^s_m \to T^r_m B$ and $\iota^{r,s}_m\colon \mathbb{D}^{r+s}_m \to \mathbb{D}^r_m \otimes \mathbb{D}^s_m$ is the canonical injection (we do not need the explicit formula for the group homomorphism of $F \circ E$). In particular, this implies $\widetilde{\mathbb{D}}^2_m = \mathbb{D}^1_m \otimes \mathbb{D}^1_m$.

Write x, x_i for the canonical coordinates on $\mathbb{D}^1_m$. In the case I, u or t is of the form $\bar{x} = x$, $y_i = kx_i$ or $\bar{x}_i = x_i$ respectively. Then the coordinate expression of $T^1_m u$ is $y_{0i} = x_i$, $y_{ij} = kx_{ij}$. This corresponds to (3) with $l = 1$. In the case II, u or t is of the form $\bar{x} = x$, $\bar{x}_i = 0$ or $y_{0i} = lx_i$, respectively. Then the coordinate expression of $T^1_m u$ is $y_i = 0$, $y_{ij} = 0$. This corresponds to (3) with $k = 0$. $\square$

Clearly, for $k = l = 1$ or $k = l = 0$ we obtain $\widetilde{J}^2_h$ or $\widetilde{J}^2_v$, respectively.

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Institute of Mathematics and Statistics
Masaryk University
Kotlářská 2, CZ 611 37 Brno
Czech Republic
E-mail: kolar@math.muni.cz