INDUCED CONNECTIONS ON TOTAL SPACES OF FIBER BUNDLES

Ivan Kolář

Abstract. We present a construction transforming a general connection Γ on a fibered manifold $Y \to M$ and a classical connection Λ on its base M into a classical connection on the total space Y by means of a vertical parallelism Φ and an auxiliary linear connection Δ. The relations to the theory of gauge-natural operators are discussed.

An important problem in the gauge theories of mathematical physics is how a principal connection Γ on a principal bundle $P \to M$ and a classical connection Λ on its base M induce a connection on the r-th principal gauge prolongation $W^r P$ of P, [1]. In [3], the authors determine all gauge-natural operators of this type. In [3] and [4], it is clarified that this result is essentially based on an exponential map on P defined by Γ and Λ. In [4] we deduced that this exponential map arises from a classical connection on the total space P that is constructed from Γ and Λ.

In the present paper, we analyze how a general connection Γ on an arbitrary fibered manifold $Y \to M$ and a classical connection Λ on M induce a classical connection on the total space Y. We clarify that one can use a vertical parallelism $\Phi\colon Y \times_M E \to VY$, where $E \to M$ is an auxiliary vector bundle and VY is the vertical tangent bundle of Y, and a linear connection Δ on $E \to M$. We write $(\Gamma, \Lambda, \Phi, \Delta)$ for the resulting classical connection on Y. Our construction covers two important special cases. The first one is the above-mentioned case of a principal connection, the second one concerns a linear connection on an arbitrary vector bundle. Both cases were discussed from the viewpoint of the theory of gauge-natural operators in [5].

In various problems concerning prolongation of connections, we realized that the torsion of the resulting connection involves important

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information about the original objects, [5]. So we characterize completely the case $(\Gamma, \Lambda, \Phi, \Delta)$ is torsion-free. In particular, we introduce the general concept of the covariant differential of a base-preserving morphism of fibered manifolds, that is based on our general theory of Lie differentation, [5]. As a special case, we obtain the concept of covariant differential $D_{(\Gamma,\Delta)}\Phi$ of Φ with respect to Γ and Δ. The main result is formulated in Proposition 6.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [5].

1. The connection $(\Gamma, \Lambda, \Phi, \Delta)$**.** Let $p\colon Y \to M$ be a fibered manifold and $q\colon E \to M$ be a vector bundle, $\dim Y = \dim E$.

Definition 1. A vertical parallelism on Y is a fibered morphism $\Phi\colon Y \times_M E \to VY$ over id_Y such that each restriction $\Phi_y\colon E_{q(y)} \to V_yY$ is a linear isomorphism of vector spaces. If $E = M \times W$ is the product bundle, then Φ is said to be of product type.

So every section $s\colon M \to E$ defines a vertical vector field $\varphi(s)\colon Y \to VY$. In the product case, Φ can be interpreted as a map $Y \times W \to VY$ and every $w \in W$ determines a vertical vector field $\varphi(w)$ on Y.

If x^i, y^p are fiber coordinates on Y, x^i, w^p are fiber coordinates on E linear on the fibers and $\eta^p = dy^p$ are the induced coordinates on VY, then the coordinate expression of Φ is

$$\eta^p = a^p_q(x, y)w^q\,. \tag{1}$$

We write $\tilde{a}^p_q$ for the inverse matrix to a^p_q.

A general connection Γ on Y can be considered either as a section $Y \to J^1Y$ or as a lifting map $Y \times_M TM \to TY$, [5]. In both cases, the coordinate expression of Γ is

$$dy^p = F^p_i(x, y)\,dx^i\,. \tag{2}$$

The equations of a linear connection Δ on E are

$$dw^p = \Delta^p_{qi}(x)w^q\,dx^i\,. \tag{3}$$

By a classical connection Λ on M we mean a linear connection on TM. Its coordinate expression is

$$d\xi^i = \Lambda^i_{jk}(x)\xi^j\,dx^k\,, \qquad \xi^i = dx^i\,. \tag{4}$$

Then the differential equations of the geodesics of Λ are

$$\frac{d^2x^i}{dt^2} = \Lambda^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}\,. \tag{5}$$

We construct the induced classical connection $\Psi = (\Gamma, \Lambda, \Phi, \Delta)$ on Y as a section $\Psi\colon TY \to J^1(TY \to Y)$. We decompose $Z \in T_yY$ into the horizontal part $hZ = \Gamma(y, Z_0)$, $Z_0 \in T_xM$, $x = p(y)$ and the vertical part $vZ = \Phi(y, Z_1)$, $Z_1 \in E_x$. We take a vector field X on M such that $j^1_x X = \Lambda(Z_0)$ and construct its Γ-lift $\Gamma X\colon Y \to TY$. Further, we consider a section s of E such that $j^1_x s = \Delta(Z_1)$.

Definition 2. For every $Z \in T_yY$, we define

$$\Psi(Z) = j^1_y\big(\Gamma X + \varphi(s)\big)\,.$$

Proposition 1. *The coordinate expression of* Ψ *is* (4) *and*

$$\begin{aligned} d\eta^p = {} & \Big(\frac{\partial F^p_i}{\partial x^j} + F^p_k \Lambda^k_{ij}\Big)\xi^i\, dx^j + \frac{\partial F^p_i}{\partial y^q}\xi^i\, dy^q \\ & + \frac{\partial a^p_r}{\partial x^j}\tilde{a}^r_q(\eta^q - F^q_i\xi^i)\, dx^j + \frac{\partial a^p_s}{\partial y^q}\tilde{a}^s_r(\eta^r - F^r_i\xi^i)\, dy^q \\ & + a^p_r \Delta^r_{sj}\tilde{a}^s_q(\eta^q - F^q_i\xi^i)\, dx^j\,. \end{aligned} \tag{6}$$

Proof. Let $\xi^i = X^i(x)$ or $w^p = s^p(x)$ be the coordinate expression of X or s, respectively. Hence

$$\frac{\partial x^i(x)}{\partial x^j} = \Lambda^i_{kj}(x) X^k(x)\,, \qquad \frac{\partial s^p(x)}{\partial x^i} = \Delta^p_{qi}(x) s^q(x)\,.$$

Then the coordinate expression of $\Gamma X + \varphi(s)$ is $\xi^i = X^i(x)$ and

$$\eta^p = F^p_i(x,y)X^i(x) + a^p_q(x,y)s^q(x)\,.$$

Differentiating this relation, we obtain (6). □

By (4) and (6), $(\Gamma, \Lambda, \Phi, \Delta)$ is a classical connection on Y that is projectable over the classical connection Λ on M.

In the case of vertical parallelism of product type, one usually considers the trivial connection on $M \times W$ with $\Delta^p_{qi} = 0$. Then we write $\Psi = (\Gamma, \Lambda, \Phi)$, cf. [4].

The following lemma generalizes Lemma 3 from [4].

Lemma 1. *Every* Γ*-lift* $\big(x^i(t), y^p(t)\big)$ *of a geodesic* $x^i(t)$ *of* Λ *is a geodesic of* $(\Gamma, \Lambda, \Phi, \Delta)$ *for arbitrary* Φ *and* Δ*.*

Proof. The Γ-lift satisfies

$$\frac{dy^p}{dt} = F^p_i\big(x(t), y(t)\big)\,\frac{dx^i}{dt}\,. \tag{7}$$

Differentiating (7) and using $x^i(t)$ is a geodesic of Λ, we obtain

$$\frac{d^2y^p}{dt^2} = \Big(\frac{\partial F^p_i}{\partial x^j} + F^p_k\Lambda^k_{ij}\Big)\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{\partial F^p_i}{\partial y^q}\frac{dx^i}{dt}\frac{dy^q}{dt}\,. \tag{8}$$

But (7) and (8) annihilates the equations of geodesics corresponding to (4) and (6) for every Φ and Δ. $\square$

2. Two important special cases. On every principal bundle $P(M,G)$ we have a canonical vertical parallelism of product type $\Pi\colon P\times\mathfrak{g}\to VP$ defined by the fundamental vector fields. For a principal connection Γ on P, we denoted $(\Gamma,\Lambda,\Pi)=N(\Gamma,\Lambda)$ in [4]. (This connection was also studied in [5] from the viewpoint of gauge-naturality.) In [4], we described all geodesics of $N(\Gamma,\Lambda)$ as follows: If $z(t)$ is a Γ-lift of a geodesic $x(t)$ of Λ and $g(t)$ is a one-parameter subgroup of G, then $z(t)\,g(t)$ is also a geodesic of $N(\Gamma,\Lambda)$. (In [4] we assumed that Λ is torsion-free, but one verifies easily that the proof remains unchanged for arbitrary Λ.)

On every vector bundle $E\to M$, we have a canonical vertical parallelism $\mathcal{V}$ determined by the well-known relation $VE=E\times_M E$. In [2], see also [5, p.410], J. Gancarzewicz constructed a classical connection $H(\Gamma,\Lambda)$ on the total space E from a linear connection Γ on $E\to M$ and a classical connection Λ on M by prescribing certain conditions on the absolute differentiation with respect to $H(\Gamma,\Lambda)$. According to [5], if

$$dy^p=\Gamma^p_{qi}(x)y^q\,dx^i \tag{9}$$

is the coordinate expression of Γ, then the equations of $H(\Gamma,\Lambda)$ are (4) and

$$d\eta^p=\Big(\frac{\partial\Gamma^p_{qi}}{\partial x^j}+\Gamma^p_{qk}\Lambda^k_{ij}-\Gamma^p_{rj}\Gamma^r_{qi}\Big)\,y^q\xi^i\,dx^j+\Gamma^p_{qi}(\xi^i\,dy^q+\eta^q\,dx^i)\,. \tag{10}$$

On the other hand, our construction yields a connection $(\Gamma,\Lambda,\mathcal{V},\Gamma)$.

Proposition 2. *We have $H(\Gamma,\Lambda)=(\Gamma,\Lambda,\mathcal{V},\Gamma)$.*

Proof. Substituting $F^p_i=\Gamma^p_{qi}y^q$, $\Delta^p_{qi}=\Gamma^p_{qi}$ and $a^p_q=\delta^p_q$ into (6), we obtain (10). $\square$

It is interesting that we can determine all geodesics even in the case of $H(\Gamma,\Lambda)$. First we deduce

Proposition 3. *If $\big(x^i(t),z^p(t)\big)$ is a Γ-lift of a geodesic $x^i(t)$ of Λ and $\big(x^i(t),y^p(t)\big)$ is an arbitrary geodesic of $H(\Gamma,\Lambda)$, then $\big(x^i(t),y^p(t)+tz^p(t)\big)$ is also a geodesic of $H(\Gamma,\Lambda)$.*

Proof. We have

$$\frac{dz^p}{dt}=\Gamma^p_{qi}\big(x(t)\big)\,z^q\frac{dx^i}{dt}\,.$$

Differentiating this relation and using (5), we obtain

$$\frac{d^2 z^p}{dt^2} = \Big(\frac{\partial \Gamma^p_{qi}}{\partial x^j} + \Gamma^p_{qk}\Lambda^k_{ij}\Big) z^q \frac{dx^i}{dt}\frac{dx^j}{dt} + \Gamma^p_{qi}\frac{dz^q}{dt}\frac{dx^i}{dt}\,.$$

Since $\big(x^i(t), y^p(t)\big)$ is a geodesic, it satisfies

$$\frac{d^2 y^p}{dt^2} = \Big(\frac{\partial \Gamma^p_{qi}}{\partial x^j} + \Gamma^p_{qk}\Lambda^k_{ij} - \Gamma^p_{rj}\Gamma^r_{qi}\Big) y^q \frac{dx^i}{dt}\frac{dx^j}{dt} + 2\Gamma^p_{qi}\frac{dy^q}{dt}\frac{dx^i}{dt}\,. \tag{11}$$

Then one verifies directly that $y^p + tz^p$ satisfies (11) as well. □

Consider an arbitrary tangent vector (ξ^i, η^p) of E at (x^i, y^p). Take the geodesic $x^i(t)$ of Λ in the direction ξ^i and construct its Γ-lift $\big(x^i(t), y^p(t)\big)$ through (x^i, y^p). We look for a Γ-lift $\big(x^i(t), z^p(t)\big)$ such that the tangent vector of $y^p(t) + tz^p(t)$ at 0 is η^p. This means $\frac{dy^p}{dt} + z^p(0) = \eta^p$. But $\frac{dy^p(0)}{dt} = \Gamma^p_{qi} y^q \xi^i$, so that our relation determines $z^p(0)$.

3. The vertical torsion. We recall that an absolute parallelism on a manifold N, $\dim N = n$, is a map $S\colon N \times \mathbb{R}^n \to TN$ such that each restriction $S(y, -)\colon \mathbb{R}^n \to T_y N$ is a linear isomorphism, [6]. Its coordinate expression is $\eta^p = a^p_q(y) w^q$. The vector fields $S(-, w)\colon N \to TN$, $w \in \mathbb{R}^n$ are called constant vector fields of S. Fixing the canonical basis of $\mathbb{R}^n$, we can interpret S as a section $\sigma\colon N \to P^1 N$ of the first order frame bundle of N. Then $\sigma(N)$ is a reduction of $P^1 N$ to the unit subgroup $\{e\}$. We have $j^1\sigma\colon N \to J^1 P^1 N$, that can be viewed as a map of $\sigma(N)$ into $J^1 P^1 N$. Using right translations, we extend $j^1\sigma$ into a principal connection Σ on $P^1 N$ that is equivalent to a classical connection on N, [5]. Direct evaluation yields that the Christoffel's of Σ are

$$\Gamma^p_{qr} = \frac{\partial a^p_s}{\partial y^r}\tilde{a}^s_q\,. \tag{12}$$

The torsion τS of S is defined to be the torsion of Σ. A classical assertion (that can be easily verified by direct evaluation) reads that S is torsion-free, iff the bracket of every two constant vector fields vanishes.

Hence a vertical parallelism Φ on Y can be viewed as a system of absolute parallelisms Φ_x on the individual fibers Y_x, $x \in M$.

Definition 3. The map

$$\tau\Phi = \bigcup_{x\in M} \tau\Phi_x\colon Y \to VY \otimes \wedge^2 V^* Y$$

is called the torsion of vertical parallelism Φ.

For $\Psi = (\Gamma, \Lambda, \Phi, \Delta)$, (4), (6) and (12) imply directly

Proposition 4. *The torsion* $\tau\Psi$ *of* Ψ *is restrictible to the fibers and the restricted map* $Y \to VY \otimes \wedge^2 V^*Y$ *concides with* $\tau\Phi$*.*

By (12), the coordinate form of $\tau\Phi = 0$ is

$$\frac{\partial a^p_s}{\partial y^r}\tilde{a}^s_q = \frac{\partial a^p_s}{\partial y^q}\tilde{a}^s_r\,. \tag{13}$$

4. Vanishing of the torsion of $(\Gamma, \Lambda, \Phi, \Delta)$**.** We characterize vanishing of the torsion $\tau\Psi$ of Ψ gradually. We write Ψ^p_{ij}, Ψ^p_{iq}, Ψ^p_{qi}, Ψ^p_{qr} for the corresponding Christoffel's of Ψ.

First we recall the general concept of Lie derivative of an arbitrary map $f\colon M \to N$ with respect to a pair of vector fields $\xi\colon M \to TM$ and $\eta\colon N \to TN$, [5]. This is the map

$$\mathscr{L}_{(\xi,\eta)} f = Tf \circ \xi - \eta \circ f\colon M \to TN\,.$$

If we consider a section $s\colon M \to Y$, its covariant differential $D_\Gamma s\colon M \to VY \otimes T^*M$ with respect to Γ satisfies

$$(D_\Gamma s)(\xi) = \mathscr{L}_{(\xi,\Gamma\xi)} s \quad \text{for every} \quad \xi\colon M \to TM\,,$$

[5]. If we have another fibered manifold $Z \to M$ with general connection Ω of the form $dz^a = G^a_i(x,z)\,dx^i$ and a base-preserving morphism $f\colon Y \to Z$, $z^a = f^a(x,y)$, then the covariant differential $D_{\Gamma,\Omega} f\colon Y \to VZ \otimes T^*M$ is defined by

$$(D_{\Gamma,\Omega} f)(\xi) = \mathscr{L}_{(\Gamma\xi,\Omega\xi)} f\,.$$

Hence its coordinate expression is

$$\frac{\partial f^a}{\partial x^i} + \frac{\partial f^a}{\partial y^p} F^p_i - G^a_i\big(x, f(x,y)\big)\,. \tag{14}$$

Consider $\Phi\colon Y \times_M E \to VY$. According to [5, p.255], Γ induces a connection $\mathcal{V}\Gamma$ on $VY \to M$ with the coordinate expression (2) and

$$d\eta^p = \frac{\partial F^p_i}{\partial y^q}\,\eta^q dx^i\,. \tag{15}$$

Further, we construct the product connection $\Gamma \times \Delta$ on $Y \times_M E$. Then $D_{\Gamma\times\Delta,\mathcal{V}\Gamma}\Phi\colon Y \times_M E \to VVY$. The values lie in a subbundle characterized by $V\pi = 0$, where $\pi\colon VY \to Y$ is the bundle projection, so that $V\pi\colon VVY \to VY$. This subbundle coincides with $VY \times_Y VY$.

Definition 4. The covariant differential $D_{(\Gamma,\Delta)}\Phi\colon Y \times_M E \to VY$ is the second component of $D_{\Gamma\times\Delta,\mathcal{V}\Gamma}\Phi$.

According to (14) and (15), its coordinate expression is

$$\Big(\frac{\partial a^p_q}{\partial x^i} + \frac{\partial a^p_q}{\partial y^r} F^r_i + a^p_r \Delta^r_{qi} - \frac{\partial F^p_i}{\partial y^r} a^r_q\Big)\, w^q\,. \tag{16}$$

By (6), the condition $\Psi^p_{qi} = \Psi^p_{iq}$ reads

$$\frac{\partial a^p_r}{\partial x^i}\tilde{a}^r_q + a^p_r \Delta^r_{si}\tilde{a}^s_q = \frac{\partial F^p_i}{\partial y^q} - \frac{\partial a^p_s}{\partial y^q}\,\tilde{a}^s_r F^r_i\,. \tag{17}$$

Then (13) and (16) imply the following assertion.

Proposition 5. *If $\tau\Phi = 0$, then $\Psi^p_{qi} = \Psi^p_{iq}$ is equivalent to $D_{(\Gamma,\Delta)}\Phi = 0$.*

Further, if $\tau\Lambda = 0$ and $\tau\Phi = 0$ and $D_{(\Gamma,\Delta)}\Phi = 0$, where $\tau\Lambda$ is the torsion of Λ, then $\Psi^p_{ij} = \Psi^p_{ji}$ is equivalent to

$$\frac{\partial F^p_i}{\partial x^j} + \frac{\partial F^p_i}{\partial y^q}\,F^q_j = \frac{\partial F^p_j}{\partial x^i} + \frac{\partial F^p_j}{\partial y^q}\,F^q_i\,. \tag{18}$$

We recall that the curvature of Γ is a map $C\Gamma\colon Y \to VY \otimes \wedge^2 T^*M$ and (18) is the coordinate form of the relation $C\Gamma = 0$, [5]. Thus we have deduced the following assertion.

Proposition 6. *The torsion of $(\Gamma, \Lambda, \Phi, \Delta)$ vanishes iff $\tau\Lambda = 0$ and $\tau\Phi = 0$ and $D_{(\Gamma,\Delta)}\Phi = 0$ and $C\Gamma = 0$.*

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Institute of Mathematics and Statistics
Masaryk University
Kotlářská 2, CZ 611 37 Brno
Czech Republic
E-mail: kolar@math.muni.cz