ON THE GAUGE VERSION OF EXPONENTIAL MAP

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Abstract. Let $P(M,G)$ be a principal bundle, Γ be a principal connection on P and Λ be a classical connection on M. For every $u \in P$, we construct a local map $\exp_u^{\Gamma,\Lambda}\colon T_uP \to P$. Then we clarify how these maps can be used for finding all natural operators transforming (Γ,Λ) into a principal connection of the r-th principal gauge prolongation W^rP of P. We also deduce that $\exp^{\Gamma,\Lambda}$ coincides with the exponential map of a classical connection on P.
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Introduction

We study certain geometric properties of principal connections on higher order principal gauge prolongations of principal bundles that are related with the gauge theories of mathematical physics, [1, 3, 4, 5]. We generalize the concept of exponential map of a classical torsion-free connection Λ on a manifold M to the case of a principal G-bundle $p\colon P \to M$. We clarify that a principal connection Γ on P and Λ define a local map $\exp_u^{\Gamma,\Lambda}\colon T_uP \to P$ for every $u \in P$. Taking into account a linear frame v on M at $p(u)$, we modify $\exp_u^{\Gamma,\Lambda}$ into a local $\mathcal{PB}$-isomorphism $\exp_{v,u}^{\Gamma,\Lambda}\colon \mathbb{R}^m \times G \to P$, $m = \dim M$. The $(r+1)$-jets of $\exp_{v,u}^{\Gamma,\Lambda}$ at $(0,e) \in \mathbb{R}^m \times G$ determine a reduction of the $(r+1)$-st principal gauge prolongation $W^{r+1}P$ to the group $G_m^1 \times G$, $G_m^1 = GL(m)$. According to our recent result, [8], this reduction defines a principal torsion-free connection $\mathcal{E}_r(\Gamma,\Lambda)$ on W^rP. In the special case G is a singleton, so that W^rP is the r-th order frame bundle P^rM of M, we reobtain the exponential operator from [7].

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Further we clarify that every reduction of W^rP to $G^1_m \times G$ identifies the adjoint bundle of W^rP with a vector bundle $\mathscr{L}^rP$ associated to $P^1M \times_M P$. Then we outline how these results can be applied to the problem of finding all naturally induced connection on W^rP. Finally, we deduce that $\exp^{\Gamma,\Lambda}$ can be interpreted as the exponential map of the classical connection $N(\Gamma,\Lambda)$ on P constructed from Γ and Λ in [6].

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from the book [9].

1. The exponential map.

By a classical connection Λ on a manifold M we mean a linear connection on TM. In the jet form, Λ is a linear morphism

$$\Lambda\colon TM \to J^1TM\,.$$

Consider a principal G-bundle $p\colon P \to M$, a principal connection Γ on P and a classical torsion-free connection Λ on M. So Λ defines a local map $\exp^\Lambda_x$ of T_xM into M for every $x \in M$. In what follows we always assume that the vectors in question lie in the domain of definition of the exponential map under consideration. For a vector $X \in T_xM$, we construct the Γ-lift $\gamma(t)$ at $u \in P$ of the curve $\exp^\Lambda_x(tX)$ on M, $x = p(u)$, and we define

$$\lambda(\Gamma,\Lambda)(X,u) = \gamma(1) \in P\,.$$

For a vector $Z \in T_uP$, we consider $Z_0 = Tp(Z) \in T_xM$ and $\omega Z \in \mathfrak{g}$, where ω is the connection form of Γ. The exponential map $\exp^G$ of G maps ωZ into an element of G and we define

$$\exp^{\Gamma,\Lambda}_u(Z) = \lambda(\Gamma,\Lambda)(Z_0,u)\exp^G(\omega Z)\,.$$

This yields a local map $\exp^{\Gamma,\Lambda}_u$ of T_uP into P. Clearly, $p \circ \exp^{\Gamma,\Lambda}_u = \exp^\Lambda_x \circ Tp$.

Lemma 1. *For every $u \in P$ and $g \in G$, the following diagram commutes*

$$\begin{array}{ccc} T_uP & \xrightarrow{\exp^{\Gamma,\Lambda}_u} & P \\ {\scriptstyle Tr_g}\downarrow & & \downarrow{\scriptstyle r_g} \\ T_{ug}P & \xrightarrow{\exp^{\Gamma,\Lambda}_{ug}} & P \end{array} \tag{1}$$

Proof. We have $\lambda(\Gamma,\Lambda)(X,ug) = \lambda(\Gamma,\Lambda)(X,u)g$ and $\omega(Tr_g(Z)) = \mathrm{Ad}(g^{-1})(\omega Z)$. For every $A \in \mathfrak{g}$, $\exp^G(\mathrm{Ad}(g^{-1})(A)) = g^{-1}\exp^G(A)g$. Hence $\exp^{\Gamma,\Lambda}_{ug}(Tr_g(Z)) = \gamma(1)g\exp^G(\mathrm{Ad}(g^{-1})(\omega Z)) = \gamma(1)\exp^G(\omega Z)g$. □

Every frame $v \in P^1_x M$ is a linear map $v\colon \mathbb{R}^m \to T_x M$. We introduce a local map $\exp^{\Gamma,\Lambda}_{v,u}$ of $\mathbb{R}^m \times G$ into P by

$$\exp^{\Gamma,\Lambda}_{v,u}(y,g) = \lambda(\Gamma,\Lambda)(v(y),u)g\,, \quad y \in \mathbb{R}^m,\ g \in G\,. \tag{2}$$

Clearly, this is a local $\mathcal{PB}$-isomorphism and we have

$$\exp^{\Gamma,\Lambda}_{vh,ug} = \exp^{\Gamma,\Lambda}_{v,u} \circ (l(h) \times l_g)\,, \quad h \in G^1_m,\ g \in G\,, \tag{3}$$

where $l(h)\colon \mathbb{R}^m \to \mathbb{R}^m$ is the linear map determined by $h \in G^1_m$ and $l_g\colon G \to G$ is the left translation.

2. The exponential operator.

Hence the rule

$$E_r(\Gamma,\Lambda)(v,u) = j^{r+1}_{(0,e)} \exp^{\Gamma,\Lambda}_{v,u} \tag{4}$$

defines a map $E_r(\Gamma,\Lambda)\colon P^1M \times_M P \to W^{r+1}P$. The structure group of $W^{r+1}P$ is $W^{r+1}_m G = G^{r+1}_m \rtimes T^{r+1}_m G = W^{r+1}_0(\mathbb{R}^m \times G)$, [9], and we have a canonical injection

$$i^{r+1}_m\colon G^1_m \times G \to W^{r+1}_m G\,, \quad i^{r+1}_m(h,g) = j^{r+1}_{(0,e)}(l(h) \times l_g)\,. \tag{5}$$

Proposition 1. *$E_r(\Gamma,\Lambda)(P^1M \times_M P)$ is a reduction of $W^{r+1}P$ to $i^{r+1}_m(G^1_m \times G)$.*

Proof. This follows directly from (3) and (5). □

By Proposition 5.2 of [8], $E_r(\Gamma,\Lambda)$ corresponds to a torsion-free connection on W^rP, that will be denoted by $\mathcal{E}_r(\Gamma,\Lambda)$. (We recall that the torsion of a connection on W^rP is defined as the covariant exterior differential of the canonical one-form $TW^rP \to \mathbb{R}^m \times \mathfrak{w}^{r-1}_m G$.) The natural operator $\mathcal{E}_r$ transforming (Γ,Λ) into a principal connection on W^rP will be called the exponential operator. In the case $G = \{pt\}$, we obtain the exponential operator of P^rM, [7].

3. The adjoint bundle of W^rP.

We write

$$\mathcal{S}^r\mathbb{R}^{m*} = \bigoplus_{k=1}^{r} S^k\mathbb{R}^{m*}\,. \tag{6}$$

For every vector space V, the (m,r)-velocities bundle $T^r_m V$ is decomposed, by means of translations, into the form $T^r_m V = V \times (T^r_m V)_0$ and a classical result reads

$$(T^r_m V)_0 = V \otimes \mathcal{S}^r\mathbb{R}^{m*}\,. \tag{7}$$

In particular, $(T^r_m\mathbb{R})_0 = \mathcal{S}^r\mathbb{R}^{m*}$. The group G^r_m acts on the right on $(T^r_mV)_0$ by the jet composition. The "linear" injection $l_r\colon G^1_m \to G^r_m$ is defined by

$$l_r(h) = j^r_0 l(h)\,.$$

Another classical result reads that the induced left action of G^1_m on $\mathcal{S}^r\mathbb{R}^{m*}$ is the standard tensorial one; we denote it by $\tau_r(h)\colon \mathcal{S}^r\mathbb{R}^{m*} \to \mathcal{S}^r\mathbb{R}^{m*}$. In the case of (7), we obtain the tensor product $\mathrm{id}_V \otimes \tau_r(h)$.

The canonical injection $\nu^r_m\colon G \to T^r_mG$ is of the form

$$\nu^r_m(g) = j^r_0\hat{g}\,,$$

where $\hat{g}$ is the constant map of $\mathbb{R}^m$ into $g \in G$. The Lie algebra $\mathfrak{t}^r_mG$ of T^r_mG coincides with $T^r_m\mathfrak{g}$. One verifies directly that $\mathrm{Ad}(j^r_0\hat{g})\colon T^r_m\mathfrak{g} \to T^r_m\mathfrak{g}$ is of the form

$$\mathrm{Ad}(g) \times \mathrm{Ad}(g) \otimes \mathrm{id}_{\mathcal{S}^r\mathbb{R}^{m*}}\colon \mathfrak{g} \times (T^r_m\mathfrak{g})_0 \to \mathfrak{g} \times (T^r_m\mathfrak{g})_0\,. \tag{8}$$

In the decomposition $W^r_mG = G^r_m \rtimes T^r_mG$, the injection $i^r_m\colon G^1_m \times G \to W^r_mG$ is of the form

$$i^r_m(h,g) = \big(l_r(h), \nu^r_m(g)\big)\,.$$

As a vector space, the Lie algebra of W^r_mG is the product

$$\mathfrak{w}^r_mG = \mathfrak{g}^r_m \times \mathfrak{t}^r_mG\,.$$

But $\mathfrak{g}^r_m \approx (T^r_m\mathbb{R}^m)_0 = \mathbb{R}^m \otimes \mathcal{S}^r\mathbb{R}^{m*}$ and $\mathfrak{t}^r_mG = T^r_m\mathfrak{g} = \mathfrak{g} \times \mathfrak{g} \otimes \mathcal{S}^r\mathbb{R}^{m*}$, so that we have

$$\mathfrak{w}^r_mG = \mathbb{R}^m \otimes \mathcal{S}^r\mathbb{R}^{m*} \times \mathfrak{g} \times \mathfrak{g} \otimes \mathcal{S}^r\mathbb{R}^{m*}\,. \tag{9}$$

The proof of the following assertion is straightforward.

Lemma 2. *For every $(A_1, A_2, A_3) \in \mathbb{R}^m \otimes \mathcal{S}^r\mathbb{R}^{m*} \times \mathfrak{g} \times \mathfrak{g} \otimes \mathcal{S}^r\mathbb{R}^{m*}$, we have*

$$\mathrm{Ad}(j^r_0 l(h), j^r_0\hat{g})(A_1, A_2, A_3) = \Big(\big(l(h) \otimes \tau_r(h)\big)(A_1), \mathrm{Ad}(g)(A_2), \big(\mathrm{Ad}(g) \otimes \tau_r(h)\big)(A_3)\Big)\,.$$

Analogously to (6), we write

$$\mathcal{S}^rT^*M = T^*M \times_M \cdots \times_M S^rT^*M\,.$$

Let $\mathrm{Ad}\,P = P[\mathfrak{g}, \mathrm{Ad}]$ be the adjoint bundle of P. We introduce

$$\mathscr{L}^rP = TM \otimes \mathcal{S}^rT^*M \times_M \mathrm{Ad}\,P \times_M \mathrm{Ad}\,P \otimes \mathcal{S}^rT^*M\,. \tag{10}$$

Clearly, $\mathscr{L}^rP$ is a vector bundle associated to $P^1M \times_M P$.

Consider an arbitrary reduction $I_r\colon P^1M \times_M P \to W^rP$ to the subgroup $i^r_m(G^1_m \times G)$. Then the results of this section can be summarized by

Proposition 2. *If we consider* $\operatorname{Ad}(W^rP)$ *as a vector bundle associated to* $I_r(P^1M \times_M P)$*, then* $\operatorname{Ad}(W^rP)$ *coincides with* $\mathscr{L}^rP$*.*

4. Application to natural operators.

The difference of two principal connections on W^rP is an arbitrary section of $\operatorname{Ad}(W^rP)\otimes T^*M$, [9]. Beside the natural connection $\mathcal{E}_r(\Gamma,\Lambda)$, our construction yields a natural reduction $E_{r-1}(\Gamma,\Lambda)(P^1M\times_M P)$ of W^rP to the group $i^r_m(G^1_m\times G)$, that identifies $\operatorname{Ad}W^rP$ with $\mathscr{L}^rP$. This implies

Proposition 3. *The natural operators transforming* (Γ,Λ) *into principal connections on* W^rP *are in bijection with the natural operators transforming* (Γ,Λ) *into sections of* $\mathscr{L}^rP\otimes T^*M$*.*

The latter operators can be determined by means of a generalization of the Utiyama theorem due to J. Janyška, [4]. Using [4] and our construction of the exponential operator, J. Janyška and J. Vondra, [5], found all natural operators transforming (Γ,Λ) into principal connections on W^rP. We remark that some further aspects of the same problem are discussed in [2].

5. Another construction of $\exp^{\Gamma,\Lambda}$.

We find remarkable that $\exp^{\Gamma,\Lambda}$ coincides with the exponential map of the classical connection $N(\Gamma,\Lambda)$ on P constructed from Γ and Λ in [6], see also Section 54.7 of [9]. We start with a slightly more general form of that construction.

Consider an arbitrary fibered manifold $p\colon Y\to M$, a general connection Γ on Y, a classical torsion-free connection Λ on M, a vector space W, $\dim W=\dim Y-\dim M$, and a vertical parallelism $\Phi\colon Y\times W\to VY$, that is an $\mathcal{FM}$-morphism over Y restricting to a linear isomorphism $W\to V_yY$ for every $y\in Y$. So every $w\in W$ defines a vertical vector field $\varphi(w)$ on Y. In this situation, we construct a classical connection $(\Gamma,\Lambda,\Phi)\colon TY\to J^1(TY\to Y)$ as follows. We decompose $X\in T_yY$ into $hX=\Gamma(y,X_0)$, $X_0\in T_xM$, $x=p(y)$, and $vX=\Phi(y,X_1)$, $X_1\in W$. We take a vector field ξ on M such that $j^1_x\xi=\Lambda(X_0)$ and consider its Γ-lift $\Gamma\xi\colon Y\to TY$. Then we define

$$(\Gamma,\Lambda,\Phi)(X)=j^1_y\big(\Gamma\xi+\varphi(X_1)\big)\,. \tag{11}$$

In fiber coordinates (x^i,y^p) on Y, we have

$$\Gamma\equiv\ dy^p=F^p_i(x,y)\,dx^i\,, \tag{12}$$

$$\Lambda\equiv\ d\xi^i=\Lambda^i_{kj}(x)\xi^k\,dx^j\,,\quad \xi^i=dx^i\,, \tag{13}$$

$$\Phi\equiv\ \eta^p=a^p_q(x,y)w^q\,,\eta^p=dy^p,\ (w^q)\in W\,. \tag{14}$$

Consider $(x_0, y_0, \xi_0, \eta_0) \in TY$. If $\xi \equiv \big(\xi^i(x)\big)$, then $\Gamma\xi \equiv \big(\xi^i(x), F^p_i(x,y)\xi^i\big)$ and $\varphi(X_1) = a^p_r(x,y)\widetilde{a}^r_q(x_0,y_0)\big(\eta^q_0 - F^q_i(x_0,y_0)\xi^i_0\big)$, where $\widetilde{a}^p_q$ means the inverse matrix to a^p_q. Passing to 1-jets, we obtain the equations of (Γ, Λ, Φ) in the form (13) and

$$\begin{aligned} d\eta^p &= \Big(\frac{\partial F^p_i}{dx^j} + F^p_k\Lambda^k_{ij}\Big)\xi^i\,dx^j + \frac{\partial F^p_i}{\partial y^q}\xi^i\,dy^q \\ &\quad + \frac{\partial a^p_r}{\partial x^j}\widetilde{a}^r_q(\eta^q - F^q_i\xi^i)\,dx^j + \frac{\partial a^p_s}{\partial y^q}\widetilde{a}^s_r(\eta^r - F^r_i\xi^i)\,dy^q\,. \end{aligned} \tag{15}$$

Lemma 3. *Every* Γ*-lift* $\big(x(t), y(t)\big)$ *of a geodesic* $x(t)$ *of* Λ *is a geodesic of* (Γ, Λ, Φ) *for arbitrary* Φ*.*

Proof. The Γ-lift satisfies

$$\frac{dy^p}{dt} = F^p_i\big(x(t), y(t)\big)\,\frac{dx^i}{dt}\,. \tag{16}$$

Differentiating (16) and using $x(t)$ is a geodesic of Λ, we find

$$\frac{d^2y^p}{dt^2} = \Big(\frac{\partial F^p_i}{dx^j} + F^p_k\Lambda^k_{ij}\Big)\frac{dx^i}{dt}\,\frac{dx^j}{dt} + \frac{\partial F^p_i}{\partial y^q}\,\frac{dx^i}{dt}\,\frac{dy^q}{dt}\,. \tag{17}$$

But (16) and (17) annihilates the equations of geodesics corresponding to (15) for every Φ. □

If P is a principal bundle, Γ is a principal connection on P, $W = \mathfrak{g}$ and Φ is the canonical vertical parallelism of P, then (Γ, Λ, Φ) coincides with $N(\Gamma, \Lambda)$ from [6]. Then our claim that $\exp^{\Gamma,\Lambda} = \exp^{N(\Gamma,\Lambda)}$ follows from the following assertion.

Proposition 4. *If* $z(t)$ *is a* Γ*-lift of a geodesic* $x(t)$ *of* Λ *and* $g(t)$ *is a one-parameter subgroup of* G*, then* $z(t)g(t)$ *is also a geodesic of* $N(\Gamma, \Lambda)$*.*

Proof. By locality and by the Ado theorem, it suffices to discuss the case $P = \mathbb{R}^m \times GL(n)$. Then we have the coordinate expressions

$$\Gamma \equiv\; dy^p_q = \Gamma^p_{ri}(x)y^r_q dx^i\,, \tag{18}$$

$$\Phi \equiv\; \eta^p_q = y^p_r w^r_q\,. \tag{19}$$

The one-parameter subgroups of $GL(n)$ are e^{At} with arbitrary $A \in \mathfrak{gl}(n)$. The Γ-lift $z(t) = \big(x^i(t), y^p_q(t)\big)$ of $x(t) = \big(x^i(t)\big)$ satisfies (18). In this situation, one verifies that $z(t)e^{At}$ is also a geodesic by direct, but tedious evaluations. □

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