Ivan Kolář

Jet-like bundles and contact elements

Author:

Prof. RNDr. Ivan Kolář, DrSc.,
Department of Mathematics and Statistics, Faculty of Science, Masaryk University, Brno, Czechia, `kolar@math.muni.cz`

ABSTRACT.

The underlying idea of this research is that the product preserving bundle functors on the category $\mathcal{M}f$ of all smooth manifolds are in bijection with the bundle functors determined by an arbitrary Weil algebra A. We describe this fact by using an original concept of A-velocity. This enables us to characterize also the product preserving bundle functors on the category $\mathcal{FM}$ of all fibered manifolds as well as the fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and local diffeomorphisms as base maps. Several geometric applications are deduced. Then we present a general introduction to the theory of semiholonomic and nonholonomic jets with the basic applications. Special attention is paid to the concept of semiholonomic contact element. We define the absolute contact differentiation and outline its application to the geometry of submanifolds of Cartan spaces.

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Contents

Preface

Using the viewpoint of algebraic geometry, A. Weil introduced an infinitely near point on a smooth manifold M as an algebra homomorphism of the algebra $C^\infty(M,\mathbb{R})$ of smooth functions on M into a local algebra A, [W]. Nowadays A is called Weil algebra and the space T^AM of the corresponding infinitely near points on M is said to be a Weil bundle. About 1985, it was deduced independently by G. Kainz and P.W. Michor, [KaMi], D.J. Eck, [Eck], and O.O. Luciano, [Lu], that the product preserving bundle functors on the category of all smooth manifolds $\mathcal{M}f$ are just the Weil functors.

This result clarified that the Weil bundles should be a good instrument for differential geometry. Hence the so-called covariant approach to Weil bundles was developed, see [KMS] or my survey article "Weil bundles as generalized jet spaces" in Handbook of Global Analysis, edited by D. Krupka and D. Saunders, Elsevier 2008, [Ko08], or [Ko16]. Under this approach, the elements of T^AM are constructed in the form of A-velocities, so that T^AM generalizes the bundle T^r_kM of (k,r)-velocities introduced by C. Ehresmann in the framework of his jet theory, [Eh]. (The original Weil's approach using the algebra homomorphisms $C^\infty(M,\mathbb{R})\to A$ is of contravariant character.) So the iteration $T^s_lT^r_k$ of two classical velocities functors is the simpliest new example of a Weil functor. In 1999, W.M. Mikulski and the author described the fiber product preserving bundle functors on the category $\mathcal{FM}_m$ of fibered manifolds with m-dimensional bases and fibered manifold morphisms with local diffeomorphisms as base maps in terms of Weil algebras, [KoMi99]. This clarified that even these functors can be viewed as a reasonable generalization of certain kinds of jet bundles.

In Chapters 1–3 of the present book we treat systematically the covariant approach to Weil bundles and we deduce several interesting geometric results in this way. We aim to differential geometry and its applications in analysis and mathematical physics. Hence we start with a rather detailed presentation of the algebraic properties of Weil algebras and their homomorphisms in Chapter 1. In accordance with our "jet-like" approach, we introduce a specific name "Weil (k, r)-algebra" for every Weil algebra of width k and order (= depth by Weil) r.

In Chapter 2, we first construct a product preserving bundle functor T^A on $\mathcal{M}f$ for every Weil algebra A by using an original concept of A-velocity. Conversely, if F is a product preserving bundle functor on $\mathcal{M}f$, then $F\mathbb{R}$ is a Weil algebra and F coincides with the Weil functor $T^{F\mathbb{R}}$. To keep the applied character of our book, we postpone the proof of this assertion to the Appendix. Further, the natural transformations $T^{A_1}M \to T^{A_2}M$ are in bijection with the algebra homomorphisms $A_1 \to A_2$ in a way, that is very suitable from the computational point of view. Special attention is paid to the flow natural exchange isomorphism $\varkappa^A_M \colon T^ATM \to TT^AM$. In particular, $\varkappa^A_M$ transforms the functorial prolongation $T^AX \colon T^AM \to T^ATM$ of every vector field $X \colon M \to TM$ into its flow prolongation $\mathcal{T}^AX \colon T^AM \to TT^AM$. Finally, we introduce the concept of contact A-element on M as the set $(\operatorname{Aut} A)(Z)$, where Z is a regular A-velocity on M. This will be used in some geometric applications in Chapters 8 and 9.

In Chapter 3 we discuss certain interesting problems from the geometry of T^A-prolongations. First we show that the Weil algebra A induces a system $L(a)_M$, $a \in A$, of natural tensor fields of type $(1, 1)$ on every bundle T^AM. Then we discuss T^A-prolongations of Lie groups and their actions. Special attentions is paid to a tensor field P of type $(1, k)$ on M. If P is antisymmetric, it is called a tangent valued k-form on M. If Q is another tangent valued l-form on M, then one can introduce the Frölicher-Nijenhuis bracket $[P, Q]$, which is a tangent valued $(k+l)$-form on M. It was clarified by L. Mangiarotti and M. Modugno, [MM84], that these forms can be applied in a deep way to the theory of general connections on arbitrary fibered manifolds. In Section 3.13 we show that there exists an infinitesimal-like algorithm for evaluating T^A-prolongations of the tangent valued forms, that is close to the synthetic differential geometry, [Kock], [MoRe], but all constructions are performed on finite dimensional manifolds.

In accordance with the "practical" intentions of the present book, the general ideas of local categories over manifolds and of bundle functors are mentioned only shortly in Sections 4.1 and 4.2. In the first part of Chapter 4 we deduce that the product preserving bundle functors on the category $\mathcal{FM}$ of fibered manifolds are of the form T^μ, i.e. they are in bijection with the Weil algebra homomorphisms $\mu \colon A \to B$. The natural transformations $T^\mu \to T^\nu$, where ν is another algebra homomorphism $\nu \colon C \to D$, are in bijection with the pairs of algebra homomorphisms $\varphi \colon A \to C$ and $\psi \colon B \to D$ satisfying $\nu \circ \varphi = \psi \circ \mu$. This can be described also in terms of fiber $(k, l; r, s)$-velocities, see Section 4.6. Further, if we consider two fibered manifolds $p\colon Y \to M$ and $\overline{p}\colon \overline{Y} \to \overline{M}$ and two $\mathcal{FM}$-morphisms f, $g\colon Y \to \overline{Y}$ with the base maps $\underline{f}$, $\underline{g}\colon M \to \overline{M}$, we say that they determine the same (q, s, r)-jet $j_y^{q,s,r} f = j_y^{q,s,r} g$ at $y \in Y$, $s \geq q \leq r$, if

$$j_y^q f = j_y^q g\,, \quad j_y^s(f \mid Y_x) = j_y^s(g \mid Y_x)\,, \quad j_x^r \underline{f} = j_x^r \underline{g}\,, \quad x = p(y)\,.$$

This concept is of fundamental importance for our general point of view to jet-like bundles.

The basic constructions in Chapter 5 are related with the product injection $i\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$, where $\mathcal{M}f_m$ is the category of all m-dimensional manifolds and their local diffeomorphisms. So every bundle functor F on $\mathcal{FM}_m$ induces a functor $F \circ i\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$. The bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ of our type are in bijection with a Weil algebra A and an action H of the r-th jet group G_m^r in dimension m on T^A: $F(M \times N)$ is the associated bundle $P^r M[T^A N, H_N]$, where $P^r M$ is the r-th order frame bundle of M. Further, F defines an equivariant natural transformation $\widetilde{t}_Y\colon J^r Y \to FY$, that is determined by an equivariant algebra homomorphism $t\colon \mathbb{D}_m^r \to A$. In Section 5.6 we deduce that the fiber product preserving bundle functors on $\mathcal{FM}_m$ are in bijection with the triples (A, H, t). If the base order of F is r, we construct a map $\psi_Y^F\colon J^r TM \times_{FTM} FTY \to TFY$ such that for a projectable vector field η on Y over a vector field ξ on M, the flow prolongation $\mathcal{F}\eta$ satisfies

$$\mathcal{F}\eta = \psi_Y^F \circ (j^r \xi \times_{\mathrm{id}\, M} F\eta)\,,$$

where $F\eta$ is the functorial prolongation of the base preserving morphism $\eta\colon Y \to TY$ over M.

Chapter 6 studies the general case of the (A, H, t)-prolongation, i.e of fiber product preserving bundle functor on $\mathcal{FM}_m$. Some specific features are reflected in the concept of weak principal bundle introduced in Section 6.3. The composition formula for two product preserving bundle functors on $\mathcal{FM}_m$, that is deduced in Section 6.6, is much more sophisticated than in the pure Weilian case of T^A and T^B. The (A, H, t)-prolongation of general connections, see Section 6.8, is a very important construction for various applications. It is remarkable that the Weil algebra technique is useful for deducing some general geometric properties of the standard jet spaces. We discuss two problems. In Section 6.9, we deduce that all smooth homomorphic images of germs of smooth maps are the r-jets for arbitrary r. In Section 6.10 we discuss the same problem for jets of foliation respecting maps.

Chapter 7 is devoted to the theory of nonholonomic and semiholonomic r-jets. Though the semiholonomic jets have much more applications, we start with some general facts about the nonholonomic case. The main theoretical reason is that Ehresmann introduced the composition of arbitrary nonholonomic r-jets, [Eh]. This enables us to define the general concept of a nonholonomic r-jet category, that is an organizing idea for the general theory. The classical approach to semiholonomic jets, presented in Section 7.4, is essentially based on the work of P. Libermann. Using an original concept of the difference tensor of semiholonomic 2-jets, we present her most famous results about principal connections on the first order frame bundle and first order G-structures in Section 7.5–7.8. Her results on higher order connections and G-structures can be only mentioned in the bibliography. The last Section 7.9 is devoted to an interesting application of the Weil algebra technique to the complete classification of all semiholonomic 3-jet categories.

Our general intention in Chapter 8 is to present some applications of our general approach to the concept of contact element. In Sections 8.1 and 8.2 we clarify some properties of the construction of iterated contact elements. In Section 8.3 we formulate the general concept of absolute contact differentiation, that is based on the idea of semiholonomic contact element. It is remarkable that the general coordinate formula for the first jet prolongation of a section of an arbitrary associated bundle from [KMS] can be applied in a very conceptual way to the coordinate formula for the absolute differentiation. In Section 8.6 we outline the coordinate approach to the iterated absolute contact differentiation in the general situation.

The most interesting application of the absolute contact differentiation is in the theory of submanifolds of Cartan spaces. A flat Cartan space is locally a Klein space. In Section 9.3 we develop a systematic approach to the Cartan method of moving frames for submanifolds of Klein spaces from the viewpoint of contact elements. In Section 9.4 we extend this procedure to the submanifolds of Cartan spaces. Our results clarify a universal character of the geometric objects in question: they are defined as the equivariant maps on the spaces of contact elements. Section 9.5 clarifies that the torsion of spaces with Cartan connection can be constructed by means of the difference tensor of semiholonomic 2-jets. Then we define the difference tensor of a semiholonomic contact 2-element, that corresponds to the reduced torsion of submanifolds of Cartan spaces. As a concrete example of the main ideas of Chapter 9 we discuss a 2-submanifold of 3-space with projective connection in Section 9.8.

The last Chapter 10 is the Appendix devoted to the proof of the fundamental theorem about product preserving bundle functors on $\mathcal{M}f$. This assertion was presented without proof in Chapter 2 as Theorem 2.4. We remark that deep results about finite order theorems were deduced by R.S. Palais with C.L. Terng, [PaTe], and C.L. Terng, [Te].

Unless otherwise specified, we use the terminology and notation from the book [KMS]. All manifolds are assumed to be Hausdorff and separable.

Chapter 1

Weil algebras

1.1 Algebras

An algebra is a vector space V together with a bilinear map $f\colon V\times V\to V$, which is called the algebra multiplication. We write $f(x,y)=xy$. The bilinearity of f implies

$$ox=o\,,\quad xo=o\,,\qquad x\in V,\ o=\text{ the zero vector of } V.$$

Let $(\overline{V},\overline{f})$ be another algebra. An algebra homomorphism $\mu\colon (V,f)\to(\overline{V},\overline{f})$ is a linear map $\mu\colon V\to\overline{V}$ preserving the multiplications. In what follows all algebras are assumed to be both commutative and associative, unless otherwise specified.
For two linear subspaces W, $Z\subset V$, we define

$$WZ=\{w_1z_1+\cdots+w_nz_n, w_i\in W, z_i\in Z\}\,, \tag{1.1}$$

where n is any integer, i.e. the elements of WZ are the finite sums of the products of an element of W and an element of Z. An ideal $I\subset V$ is a linear subspace such that

$$xa\in I\quad\text{for all}\quad x\in I,\ a\in V\,.$$

For every subset $S\subset V$, we denote by $\langle S\rangle$ the ideal generated by S, i.e. the smallest ideal in V containing S. The factor vector space V/I is an algebra with respect to the multiplication

$$(a+I)\,(b+I)=ab+I\,.$$

On the other hand, the kernel of every algebra homomorphism is an ideal.

An element $a \in V$ is said to be nilpotent, if $a^k = o$ for some integer k. The set N of all nilpotent elements of V is an ideal. Indeed, $n_1^r = o$ and $n_2^s = o$ implies $(n_1 + n_2)^{r+s} = o$. Further, $n^r = o$ implies $(an)^r = a^r n^r = o$ for every $a \in A$.
A unit of V is an element $e \neq o$ satisfying $ex = x$ for all $x \in V$. If the unit exists, it is unique. An algebra with unit is said to be unital. The unit defines an injection $\mathbb{R} \hookrightarrow V$, $c \mapsto ce$. In this situation, we write $\mathbb{R} \subset V$ and we identify e with $1 \in \mathbb{R}$ and the zero vector o with $0 \in \mathbb{R}$. The homomorphisms of unital algebras $V \to \overline{V}$ are assumed to transform the unit of V into the unit of $\overline{V}$.

1.2 Weil algebras

The following concept was called local algebra in the original paper by A. Weil, [W].

Definition. A Weil algebra A is a finite dimensional, commutative, associative and unital algebra of the form

$$A = \mathbb{R} \times N\,, \tag{1.2}$$

where N is the ideal of all nilpotent elements of A.

In particular, $\mathbb{R}$ is a trivial Weil algebra with $N = 0$. Let $\overline{A} = \mathbb{R} \times \overline{N}$ be another Weil algebra and $\mu\colon A \to \overline{A}$ be a homomorphism. Then the restriction and corestriction of μ to $\mathbb{R} \subset A$ and $\mathbb{R} \subset \overline{A}$ is the identity and μ transforms N into $\overline{N}$. The zero homomorphism $\mathcal{O}\colon A \to \overline{A}$ maps N into $0 \in \overline{A}$. We write $\operatorname{Hom}(A, \overline{A})$ for the set of all algebra homomorphisms of A into $\overline{A}$. The category of Weil algebras and their homomorphisms will be denoted by $\mathcal{W}ei$.

1.3 The invertible elements

An ideal $I \subset A$ is called maximal, if $I \neq A$ and there exists no ideal $J \subsetneqq A$ such that $I \subsetneqq J$. First we deduce that N is the unique maximal ideal. Indeed, let I be an ideal satisfying $N \subsetneqq I \subset A$. Then $\varrho + n \in I$ for some $0 \neq \varrho \in \mathbb{R}$ and $n \in N$. Hence $\varrho \in I$ and $I = A$.

Proposition. *All invertible elements of A are of the form $\varrho + n$, $0 \neq \varrho \in \mathbb{R}$, $n \in N$.*

Proof. If $\varrho + n$ were not invertible, the set $B = \{a(\varrho + n), a \in A\}$ would be an ideal different from A. Hence $B \subset N$, so that $\varrho + n \in N$. Then $0 \neq \varrho = \big((\varrho + n) - n\big) \in N$, which is a contradiction. □

1.4 The order of A

We define N^r by the iteration $N^1 = N$, $N^r = N(N^{r-1})$.

Proposition. *There exists an integer r such that $N^r = 0$.*

Proof. We have a non increasing sequence of vector spaces

$$N \supset N^2 \supset \cdots \supset N^s \supset N^{s+1} \supset \dots$$

Since N is finite dimensional, there exists an integer r such that

$$N^r = N^{r+1} = N(N^r)\,. \tag{1.3}$$

Let $a_1, \dots, a_k$ be a linear basis of N^r. By (1.3),

$$a_i = \sum_{j=1}^{k} z_{ij} a_j\,, \qquad z_{ij} \in N,\ i = 1, \dots, k\,. \tag{1.4}$$

This can be rewritten as a system of linear equations

$$\sum_{j=1}^{k} \big(\delta_{ij} - z_{ij}\big) a_j = 0\,.$$

The determinant $\det(\delta_{ij} - z_{ij})$ is of the form $1 + z'$, $z' \in N$, so that it is invertible. By the Cramer rule, $a_i = 0$ for all i. Hence $N^r = 0$. □

The smallest r with the property $N^{r+1} = 0$ is called *the order* $\operatorname{ord} A$ of A. (We remark that A. Weil used the term “depth”.)

1.5 The width of A

The dimension wA of the vector space N/N^2 is called *the width* of A. A Weil algebra of width k and order r will be said to be a *Weil* (k, r)*-algebra*. Every algebra homomorphism $\mu\colon A \to B = \mathbb{R} \times Q$ induces a linear map

$$\widetilde{\mu}\colon N/N^2 \to Q/Q^2\,, \quad \widetilde{\mu}(a + N^2) = \mu(a) + Q^2\,, \quad a \in N\,. \tag{1.5}$$

It is useful to deduce that (1.5) is of functorial character.
Consider three Weil algebras $\mathbb{R} \times N$, $\mathbb{R} \times Q$, $\mathbb{R} \times P$ and two Weil algebra homomorphism, given on the nilpotent parts $f\colon N \to Q$ and $g\colon Q \to P$. We know $f(N) \subset Q$ and $g(Q) \subset P$. For $(n + N^2) \in N/N^2$, we define

$$\widetilde{f}(n + N^2) = \bigl(f(n) + Q^2\bigr) \in Q/Q^2 \,. \tag{1.6}$$

Further, $\widetilde{g}\colon Q/Q^2 \to P/P^2$ is given by

$$\widetilde{g}(h + Q^2) = \bigl(g(h) + P^2\bigr) \in P/P^2 \,,$$

$h \in P$. Hence $\widetilde{g} \circ \widetilde{f}\colon N/N^2 \to P/P^2$ satisfies

$$(\widetilde{g} \circ \widetilde{f})(n + N^2) = \widetilde{g}\bigl(f(n) + Q^2\bigr) = \widetilde{g}\bigl(\widetilde{f}(n + N^2)\bigr) \,,$$

so that $\widetilde{\ }$ is a functor from the category $\mathcal{W}\!ei$ into the category of vector spaces.

1.6 The algebra $\mathbb{D}_k^r$

Let $\mathbb{R}[x_1, \dots, x_k]$ be the algebra of all polynomials in k undetermined. The simpliest example of a Weil algebra is

$$\mathbb{D}_k^r = \mathbb{R}[x_1, \dots, x_k]/\langle x_1, \dots, x_k\rangle^{r+1} \,. \tag{1.7}$$

As a vector space, $\mathbb{D}_k^r$ is the set of all polynomials of degree at most r in k undetermined with the standard addition and multiplication by real scalars. The product of P, $Q \in \mathbb{D}_k^r$ is the "truncated" one: we multiply P and Q as polynomials and we neglect the terms of degree higher than r. Clearly, $\operatorname{ord}(\mathbb{D}_k^r) = r$ and $w(\mathbb{D}_k^r) = k$.
We shall write N_k^r for the nilpotent part of $\mathbb{D}_k^r$. The elements of N_k^r are the polynomials without absolute terms.
In particular, $\mathbb{D}_1^1$ is the classical algebra $\mathbb{D}$ of dual (or Study) numbers. Its elements can be written as $a + be$, a, $b \in \mathbb{R}$, with e satisfying $e^2 = 0$.

1.7 A as a factor algebra

The following assertion gives a rather concrete description of Weil algebras.

Proposition. *Every Weil algebra A of order $\leq r$ and of width k is a factor algebra of $\mathbb{D}_k^r$.*

Proof. Choose $a_1, \dots, a_k \in N$ such that

$$a_1 + N^2, \dots, a_k + N^2 \tag{1.8}$$

is a basis of the vector space N/N^2. Define $\pi\colon \mathbb{D}^r_k \to A$, $\pi(P) = P(a_1, \dots, a_k)$. First we deduce that π is a homomorphism, i.e. $(PQ)(a_1, \dots, a_k) = P(a_1, \dots, a_k)\, Q(a_1, \dots, a_k)$. Indeed, PQ on the left hand side is the product in $\mathbb{D}^r_k$, while $P(a_1, \dots, a_k)\, Q(a_1, \dots, a_k)$ on the right hand side is the standard product of polynomials in $a_1, \dots, a_k$, in which the condition $N^{r+1} = 0$ suppresses the terms of degree $> r$.
It remains to show that π is surjective. Since (1.8) is a basis, for every $a \in N$ we have

$$a + N^2 = c_1(a_1 + N^2) + \dots + c_k(a_k + N^2)\,, \quad c_1, \dots, c_k \in \mathbb{R}\,,$$

i.e. $a - c_1 a_1 - \dots - c_k a_k = n_1 \in N^2$. One verifies directly that the elements $a_i a_j + N^3$ generate linearly N^2/N^3, so that there exist $c_{ij} \in \mathbb{R}$ such that

$$n_1 = \sum_{i,j=1}^{k} c_{ij} a_i a_j + n_2\,, \qquad n_2 \in N^3\,.$$

In the $(l-1)$-st step of such procedure, we obtain

$$n_{l-1} = \sum_{i_1, \dots, i_l = 1}^{k} c_{i_1 \dots i_l} a_{i_1} \dots a_{i_l} + n_l\,, \qquad n_l \in N^{l+1}\,.$$

But $N^{r+1} = 0$, so that after r steps we have

$$a = \sum_{i=1}^{k} c_i a_i + \dots + \sum_{i_1, \dots, i_r = 1}^{k} c_{i_1 \dots i_r} a_{i_1} \dots a_{i_r}\,.$$

The c's determine a polynomial $P \in N^r_k$ satisfying $a = P(a_1, \dots, a_k)$. □

1.8 Reparametrizations

We shall use heavily the following interpretation of $\mathbb{D}^r_k$ in terms of jets. By (1.7), the elements of $\mathbb{D}^r_k$ are r-jets of functions on $\mathbb{R}^k$ at 0, i.e. $\mathbb{D}^r_k = J^r_0(\mathbb{R}^k, \mathbb{R})$. The addition in $\mathbb{D}^r_k$, the multiplication by reals and the multiplication in $\mathbb{D}^r_k$ are expressed by the formulae

$$j^r_0\gamma + j^r_0\delta = j^r_0(\gamma + \delta)\,, \quad c j^r_0\gamma = j^r_0(c\gamma)\,, \quad (j^r_0\gamma)(j^r_0\delta) = j^r_0(\gamma\delta)\,,$$

$\gamma, \delta\colon \mathbb{R}^k \to \mathbb{R}$, $c \in \mathbb{R}$.
We denote the composition of jets by the same symbol $\circ$ as the composition of maps.
Every r-jet $X \in J_0^r(\mathbb{R}^l, \mathbb{R}^k)_0$ induces an algebra homomorphism $\mathbb{D}_k^r \to \mathbb{D}_l^r$ by

$$Y \mapsto Y \circ X\,, \quad Y \in \mathbb{D}_k^r = J_0^r(\mathbb{R}^k, \mathbb{R})\,. \tag{1.9}$$

Geometrically, (1.9) is a reparametrization of the elements of $\mathbb{D}_k^r$.

Proposition. *We have* $\mathrm{Hom}\,(\mathbb{D}_k^r, \mathbb{D}_l^r) = J_0^r(\mathbb{R}^l, \mathbb{R}^k)_0$.

Proof. Let $\mu\colon \mathbb{D}_k^r \to \mathbb{D}_l^r$ be an algebra homomorphism. Then $\mu(x_i) = P_i \in N_k^r$ for every $i = 1, \dots, k$. Hence $P = (P_1, \dots, P_k)$ is a k-tuple of polynomials of degree at most r without the absolute term in l undetermined. This defines an r-jet $P \in J_0^r(\mathbb{R}^l, \mathbb{R}^k)_0$. By the definition of the jet composition, $\mu(Y) = Y \circ P$ for all $Y \in \mathbb{D}_k^r$. □

1.9 Weil (k, r)-algebras

Definition. A Weil algebra of width k and order r is called a Weil (k, r)-algebra.

Proposition. *Let A be a Weil (k, r)-algebra and π, $\varrho\colon \mathbb{D}_k^r \to A$ be two surjective algebra homomorphisms. Then there exists an algebra isomorphism $\sigma\colon \mathbb{D}_k^r \to \mathbb{D}_k^r$ satisfying $\pi = \varrho \circ \sigma$.*

Proof. Let x_i be the basis of $\mathbb{D}_k^r$. Write $a_i = \pi(x_i)$ and choose some $P_i \in \mathbb{D}_k^r$ satisfying $\varrho(P_i) = a_i$. Consider the homomorphism $\sigma\colon \mathbb{D}_k^r \to \mathbb{D}_k^r$ transforming x_i into P_i. Then $\varrho(\sigma(x_i)) = \pi(x_i)$, so that $\pi = \varrho \circ \sigma$. Consider

$$\widetilde{\pi}, \widetilde{\varrho}\colon N_k^r/(N_k^r)^2 \to N/N^2,\ \widetilde{\sigma}\colon N_k^r/(N_k^r)^2 \to N_k^r/(N_k^r)^2\,.$$

Since both π and ϱ are surjective algebra morphisms, $\widetilde{\pi}$ and $\widetilde{\varrho}$ are surjective linear maps. By Proposition 1.5, $\widetilde{\pi} = \widetilde{\varrho} \circ \widetilde{\sigma}$. But σ is determined by a reparametrization $Y \mapsto Y \circ X$, $Y \in \mathbb{D}_k^r$, $X \in J_0^r(\mathbb{R}^k, \mathbb{R}^k)_0$. Hence X is an invertible r-jet, so that σ is an isomorphism. □

1.10 Algebra homomorphisms

Let A and B be two Weil algebras with $k = wA$, $l = wB$, $r = \max(\operatorname{ord} A, \operatorname{ord} B)$ and $\mu\colon A \to B$ be an algebra homomorphism. Consider two surjective homomorphisms $\pi\colon \mathbb{D}_k^r \to A$ and $\varrho\colon \mathbb{D}_l^r \to B$. According to

Section 1.5, one deduces there is an algebra homomorphism $\sigma\colon \mathbb{D}_k^r \to \mathbb{D}_l^r$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{D}_k^r & \xrightarrow{\sigma} & \mathbb{D}_l^r \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ A & \xrightarrow{\mu} & B \end{array} \tag{1.10}$$

An element $X \in J_0^r(\mathbb{R}^l, \mathbb{R}^k)_0$ such that (1.10) with $\sigma = X$ commutes will be called μ-admissible.
By Proposition 1.7, A can be expressed as a factor algebra

$$A = \mathbb{D}_k^r / I\,. \tag{1.11}$$

Let $P_h(x_1, \dots, x_k)$ be some generators of ideal I, $h = 1, \dots, s$. To determine all algebra homomorphisms $A \to B$, we first consider an arbitrary k-tuple b_i of elements in the nilpotent part of B to be the images of $\pi(x_i)$. Then the rule $b_i = \mu\big(\pi(x_i)\big)$ generates an algebra homomorphism $\mu\colon A \to B$, if and only if $P_h(b_1, \dots, b_k) = 0$ for all h. An example can be found in Section 1.14 below.

1.11 Automorphisms

The group $\operatorname{Aut} A$ of all algebra automorphisms of A is a closed subgroup in the group $GL(A)$ of all linear automorphisms of A, so a Lie group. By Section 1.8, the group $\operatorname{Aut}(\mathbb{D}_k^r)$ coincides with the jet group $G_k^r = \operatorname{inv} J_0^r(\mathbb{R}^k, \mathbb{R}^k)_0$ of all invertible r-jets of $\mathbb{R}^k$ into $\mathbb{R}^k$ with source and target 0.
A derivation of A is a linear map $f\colon A \to A$ satisfying $f(a_1 a_2) = f(a_1)a_2 + a_1 f(a_2)$ for all $a_1, a_2 \in A$. According to the general theory, the Lie algebra $\mathfrak{Aut}A$ of $\operatorname{Aut} A$ coincides with the Lie algebra $\operatorname{Der} A$ of all derivations of A.

1.12 Sums

For two Weil algebras $A = \mathbb{R} \times N_A$, $B = \mathbb{R} \times N_B$, the vector space

$$A \oplus B = \mathbb{R} \times N_A \times N_B \tag{1.12}$$

is also a Weil algebra with respect to the multiplication

$$(c_1, a_1, b_1)(c_2, a_2, b_2) = (c_1 c_2, c_1 a_2 + c_2 a_1 + a_1 a_2, c_1 b_2 + c_2 b_1 + b_1 b_2)\,,$$

$c_i \in \mathbb{R}$, $a_i \in N_A$, $b_i \in N_B$, $i = 1, 2$. One can say that $A \oplus B$ is the sum of A and B.

If A is expressed by (1.11) and B analogously by

$$\mathbb{R}[y_1, \ldots, y_l]/J\,, \tag{1.13}$$

then $\langle I, J, x_i y_p \rangle$, $i = 1, \ldots, k$, $p = 1, \ldots, l$ is an ideal in $\mathbb{R}[x_1, \ldots, x_k, y_1, \ldots, y_l]$ and we have

$$A \oplus B = \mathbb{R}[x_1, \ldots, y_l]/\langle I, J, x_i y_p \rangle\,.$$

In particular, this implies

$$w(A \oplus B) = wA + wB\,, \quad \operatorname{ord}(A \oplus B) = \max(\operatorname{ord} A, \operatorname{ord} B)\,.$$

1.13 Tensor products

In general, the tensor product $V_1 \otimes V_2$ of two algebras (V_1, f_1) and (V_2, f_2) is also an algebra, whose multiplication f is the tensor product of f_1 and f_2. Thus, for the decomposable tensors $v_1 \otimes v_2$, $\overline{v}_1 \otimes \overline{v}_2$, we have

$$f(v_1 \otimes v_2, \overline{v}_1 \otimes \overline{v}_2) = f_1(v_1, \overline{v}_1) \otimes f_2(v_2, \overline{v}_2)\,.$$

In the case of two Weil algebras $A = \mathbb{R} \times N_A$, $B = \mathbb{R} \times N_B$, $A \otimes B$ is also a Weil algebra with the nilpotent part $N_A \times N_B \times N_A \otimes N_B$. If A is expressed by (1.11) and B by (1.13), then $\langle I, J \rangle$ is an ideal in $\mathbb{R}[x_1, \ldots, x_k, y_1, \ldots, y_l]$ and we have

$$A \otimes B = \mathbb{R}[x_1, \ldots, y_l]/\langle I, J \rangle\,. \tag{1.14}$$

In particular, this implies,

$$w(A \otimes B) = wA + wB\,, \quad \operatorname{ord}(A \otimes B) = \operatorname{ord} A + \operatorname{ord} B\,.$$

For instance, $\mathbb{D} \otimes \mathbb{D}$ is of the form

$$\mathbb{R}[x, y]/\langle x^2, y^2 \rangle\,. \tag{1.15}$$

1.14 Example

We determine all algebra homomorphisms $\mu \colon \mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$. Write

$$\mu(x) = c_1 x + c_2 y + c_3 xy\,, \quad \mu(y) = c_4 x + c_5 y + c_6 xy\,.$$

The conditions $\big(\mu(x)\big)^2 = 0$ and $\big(\mu(y)\big)^2 = 0$ imply $c_1 c_2 = 0$ and $c_4 c_5 = 0$. By Section 1.10, all algebra homomorphisms $\mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$ form the following 4 four-parameter families

$$c_1 = 0 = c_4\,, \quad c_2 = 0 = c_4\,, \quad c_1 = 0 = c_5\,, \quad c_2 = 0 = c_5$$

with arbitrary c_3 and c_6.

1.15 Weil algebras described in terms of germs

Sometimes it is suitable to express a Weil algebra as a factor algebra of the algebra $\mathcal{E}(k)$ of germs of smooth functions on $\mathbb{R}^k$ at 0. Let $\mathfrak{m}(k) \subset \mathcal{E}(k)$ be the ideal of all germs with value 0. Formula (1.7) can be rewritten as

$$\mathbb{D}^r_k = \mathcal{E}(k)/\mathfrak{m}(k)^{r+1}\,. \tag{1.16}$$

Write

$$\varphi\colon \mathcal{E}(k) \to \mathbb{D}^r_k$$

for the factor projection.

Let A be a Weil (k,r)-algebra and $\pi\colon \mathbb{D}^r_k \to A$ be a surjective algebra homomorphism. If we define

$$\mathcal{I}_A = \big\{a \in \mathcal{E}(k), \pi(\varphi(a)) \in A\big\}\,,$$

then we have

$$A = \mathcal{E}(k)/\mathcal{I}_A\,. \tag{1.17}$$

This implies that the Weil algebras can be considered as factor algebras of $\mathcal{E}(k)$ by the ideals of finite codimension different from $\mathcal{E}(k)$.

Chapter 2

Weil bundles

2.1 A-velocities

Our covariant approach to Weil bundles is based on the concept of A-velocity. This generalizes the classical concept of (k, r)-velocity by C. Ehresmann. We recall that the classical construction of (k, r)-velocities is a bundle functor T^r_k on $\mathcal{M}f$ defined by

$$T^r_k M = J^r_0(\mathbb{R}^k, M)\,, \quad T^r_k f(j^r_0\gamma) = j^r_0(f \circ \gamma)\,, \qquad \gamma\colon \mathbb{R}^k \to M\,, \tag{2.1}$$

for every manifold M and every smooth map $f\colon M \to N$.
Consider a Weil (k, r)-algebra A together with an algebra homomorphism $\pi\colon \mathbb{D}^r_k \to A$ from Section 1.9. By that section, π is determined up to an isomorphism $\mathbb{D}^r_k \to \mathbb{D}^r_k$, so that the following definition is independent of π. Let M be a manifold.

Definition. Two maps γ, $\delta\colon \mathbb{R}^k \to M$ determine the same A-velocity $j^A\gamma = j^A\delta$, if for every smooth function $\varphi\colon M \to \mathbb{R}$

$$\pi\big(j^r_0(\varphi \circ \gamma)\big) = \pi\big(j^r_0(\varphi \circ \delta)\big)\,. \tag{2.2}$$

Proposition. *Let $\gamma^i(t_1, \dots, t_k)$ or $\delta^i(t_1, \dots, t_k)$ be the coordinate expression of γ or δ. Then $j^A\gamma = j^A\delta$ if and only if $j^A\gamma^i = j^A\delta^i$ for all $i = 1, \dots, \dim M$.*

Proof. By locality, we may assume $M = \mathbb{R}^m$ and consider $\varphi^i\colon \mathbb{R}^m \to \mathbb{R}$, $\varphi^i(x_1, \dots, x_m) = x_i$. Then (2.2) implies $j^A\gamma^i = j^A\delta^i$. The converse assertion can be deduced by the inverse procedure. □

Let $T^A M$ be the set of all A-velocities on M.

2.2 Weil functors

Proposition 2.1 implies

$$T^A\mathbb{R} = \pi(\mathbb{D}_k^r) = A \quad \text{and} \quad T^A\mathbb{R}^m = A^m .$$

So $T^AM \to M$ is a fibered manifold that is called a Weil bundle. For a smooth map $f\colon M \to N$, we define

$$T^Af\colon T^AM \to T^AN \quad \text{by} \quad T^Af(j^A\gamma) = j^A(f\circ\gamma) . \tag{2.3}$$

To demonstrate the correctness of (2.3), it suffices to discuss the case $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$. Then $f = (f_1,\ldots,f_n)$, $f_i\colon \mathbb{R}^m \to \mathbb{R}$, so that $T^Af(j^A\gamma) = (T^Af_1(j^A\gamma),\ldots,T^Af_n(j^A\gamma))$, which is independent of the choice of $j^A\gamma = j^A\delta$ by Proposition 2.1.
For another smooth map $f\colon N \to Q$, we have

$$T^A(g\circ f)(j^A\gamma) = T^Ag\big(T^Af(j^A\gamma)\big) .$$

Hence we obtain a bundle functor T^A on $\mathcal{M}f$ called Weil functor. Clearly, $T^{\mathbb{D}_k^r} = T_k^r$, so that $T^{\mathbb{D}}$ is the tangent functor T. Our construction yields a surjective map $\pi_M\colon T_k^rM \to T^AM$ such that the following diagram commutes for every $f\colon M \to N$

$$\begin{array}{ccc} T_k^rM & \xrightarrow{T_k^rf} & T_k^rN \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ T^AM & \xrightarrow{T^Af} & T^AN \end{array} \tag{2.4}$$

A section $M \to T^AM$ is said to be an A-field on M.

2.3 The contravariant approach

Definition 2.1 is a modification of what is called the covariant approach to Weil functors in [KMS]. By Section 1.15, A can be viewed as a factor algebra $A = \mathcal{E}(k)/\mathcal{I}$, where $\mathcal{I}$ is the ideal determined by I in $\mathcal{E}(k)$. Then two maps γ, $\delta\colon \mathbb{R}^k \to M$ satisfy $j^A\gamma = j^A\delta$, if and only if $\varphi\circ\gamma - \varphi\circ\delta \in \mathcal{I}$ for every germ φ of smooth function on M at $x = \gamma(0) = \delta(0)$.
The original ideas by A. Weil, [W], were inspired by the algebraic geometry. So his approach is of contravariant character. All smooth functions on M form an algebra $C^\infty(M,\mathbb{R})$ with respect to the pointwise multiplication.

Weil defined a so-called infinitely near point of type A on M as an algebra homomorphism

$$C^\infty(M,\mathbb{R}) \to A\,.$$

We show that the set of all algebra homomorphisms $\mathrm{Hom}\,(C^\infty(M,\mathbb{R}),A)$ is canonically isomorphic to $T^A M$.
For every $f \in C^\infty(M,\mathbb{R})$ and every $j^A\gamma \in T^A M$, we define

$$(j^A\gamma)(f) = j^A(f\circ\gamma) \in A\,. \tag{2.5}$$

This is an algebra homomorphism. Indeed, $(j^A\gamma)(f_1+f_2) = j^A(f_1\circ\gamma+f_2\circ\gamma)$ and $(j^A\gamma)(f_1f_2) = j^A((f_1\circ\gamma)(f_2\circ\gamma))$. Since our operations behave well with respect to localization, it suffices to consider the case $M = \mathbb{R}^m$, so that $T^A\mathbb{R}^m = A^m$. Then Proposition 2.1 implies that (2.5) establishes a bijection between $T^A\mathbb{R}^m$ and $\mathrm{Hom}\,(C^\infty(\mathbb{R}^m,\mathbb{R}),A)$. Further, every smooth map $f\colon M_1 \to M_2$ induces a canonical algebra homomorphism $(C^\infty M_2,\mathbb{R}) \to (C^\infty M_1,\mathbb{R})$, which maps every $\mathrm{Hom}\,(C^\infty M_2,A)$ into $\mathrm{Hom}\,(C^\infty M_1,A)$.

2.4 Product preserving bundle functors

Consider the product $M_1 \xleftarrow{p_1} M_1\times M_2 \xrightarrow{p_2} M_2$ of two manifolds together with the product projections. A bundle functor F on $\mathcal{M}f$ is said to be product preserving, if $F(M_1\times M_2) = FM_1\times FM_2$. More precisely, this means that

$$FM_1 \xleftarrow{Fp_1} F(M_1\times M_2) \xrightarrow{Fp_2} FM_2$$

is also a product. Every Weil functor preserves products. Indeed, if $j^A\gamma_1 \in T^AM_1$ and $j^A\gamma_2 \in T^AM_2$, $\gamma_1\colon\mathbb{R}^k\to M_1$, $\gamma_2\colon\mathbb{R}^k\to M_2$, then $(\gamma_1,\gamma_2)\colon\mathbb{R}^k\to M_1\times M_2$ and $j^A(\gamma_1,\gamma_2) = (j^A\gamma_1, j^A\gamma_2)$.
The converse assertion is a fundamental theoretical result. Let F be a product preserving bundle functor on $\mathcal{M}f$. Write *pt* for one point set and $i_x\colon pt\to M$ for the map $i_x(pt) = x$, $x\in M$. Since F preserves products, we have $F(pt) = pt$. A natural injection $\nu^F_M\colon M\to FM$ is defined by

$$\nu^F_M(x) = Fi_x(pt)\,. \tag{2.6}$$

Applying F to the addition $a\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and the multiplication $m\colon\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ of reals, we obtain

$$Fa\colon F\mathbb{R}\times F\mathbb{R}\to F\mathbb{R}\,,\qquad Fm\colon F\mathbb{R}\times F\mathbb{R}\to F\mathbb{R}\,.$$

One verifies easily that $F\mathbb{R}$ with the addition Fa and the multiplication by real scalars $ca = Fm(\nu_{\mathbb{R}}(c), a)$, $c \in \mathbb{R}$, $a \in F\mathbb{R}$ is a vector space. The proof of the following assertion can be found in the Appendix.

Theorem. *$F\mathbb{R}$ is a Weil algebra with respect to the multiplication Fm and F coincides with the Weil functor $T^{F\mathbb{R}}$.*

A simple example of a bundle functor on $\mathcal{M}f$ that does not preserve products is the second tensor power of the tangent functor T. Indeed, $\dim \overset{2}{\otimes} T(M \times N) > \dim \overset{2}{\otimes} TM + \dim \overset{2}{\otimes} TN$ provided $\dim M > 0$ and $\dim N > 0$.
Another interesting example of such functor is the r-th order tangent vector bundle $T^{(r)}$. Clearly, $J^r(M, \mathbb{R})_0$ is a vector bundle over M and we define

$$T^{(r)}M = J^r(M, \mathbb{R})_0^*$$

as the dual vector bundle. For every map $f\colon M \to N$, the jet composition $X \mapsto X \circ (j^r_x f)$, $x \in M$, $X \in (T_1^{r*}N)_{f(x)}$ defines a linear map

$$\lambda(j^r_x f)\colon (T_1^{r*}N)_{f(x)} \to (T_1^{r*}M)_x\,.$$

The dual maps

$$\big(\lambda(j^r_x f)\big)^*\colon (T_1^{r*}M)_x^* \to (T_1^{r*}N)^*_{f(x)}$$

determine a bundle functor $T^{(r)}$ on $\mathcal{M}f$ with values in the category of vector bundles. For $r > 1$, these functors do not preserve products by the dimension argument.
In what follows, we shall simplify the notation to

$$\nu^A_M\colon M \to T^A M \tag{2.7}$$

for the map (2.6) with $F = T^A$.

2.5 The coordinate expression of $T^A f$

The Taylor expansion of order r with remainder of the components of a map $f\colon \mathbb{R}^m \to \mathbb{R}^n$, $y^p = f^p(x^i)$ is

$$f^p(x^i) = \sum_{|\alpha| \le r} \frac{1}{\alpha!} D_\alpha f^p(x^i) z^\alpha + \sum_{|\beta| = r+1} \frac{1}{\beta!} D_\beta f^p(\xi^i) z^\beta\,, \tag{2.8}$$

where $\xi^i \in \mathbb{R}^m$, α and β are multiindexes of range m, $z^\alpha = (z^1)^{\alpha_1} \dots (z^m)^{\alpha_m}$ and analogously z^β. Write $a^i = x^i + n^i$ for the i-th component in A^m,

$n^i \in N_A$, and b^p for the p-th component in A^n. Since $z^i \mid 0 = 0$, we have $T^A z^i = n^i$. Applying T^A to (2.8) with fixed x^i and using the fact that T^A preserves products, we obtain

$$b^p = f^p(x^i) + \sum_{|\alpha| \leq r} \frac{1}{\alpha!} D_\alpha f^p(x^i) n^\alpha \tag{2.9}$$

where $n^\alpha = (n^1)^{\alpha_1} \dots (n^m)^{\alpha_m}$ with the multiplication in A and $(n^1)^{\beta_1} \dots (n^m)^{\beta_m} = 0$ for $\beta_1 + \dots \beta_m = r + 1$ because of A is of the order r.

In particular, if $f \colon \mathbb{R}^m \to \mathbb{R}$ is a real valued function, then (2.9) with no superscript p expresses the A-valued function $T^A f \colon T^A \mathbb{R}^m \to A$. We shall need the explicit formula in the simpliest case $A = \mathbb{D}$. Write x_1^i for the additional coordinates on $T\mathbb{R}^m$. Then the coordinate form of $Tf \colon T\mathbb{R}^m \to \mathbb{D}$ is

$$f(x^i) + e\Big(\frac{\partial f}{\partial x^i} x_1^i\Big) \,. \tag{2.10}$$

Example. Consider the iterated tangent functor TT. The elements of its Weil algebra $\mathbb{D} \otimes \mathbb{D} = \mathbb{R}[t, \tau] / \langle t^2, \tau^2 \rangle$ are of the form $x + ut + vt + wt\tau$. The corresponding coordinates on $TT\mathbb{R}^m = (\mathbb{D} \otimes \mathbb{D})^m$ are x^i, u^i, v^i, w^i. From the classical point of view, x^i are the "original" coordinates on $\mathbb{R}^m$, the role of the "first order" coordinates u^i and v^i is more or less symmetric and w^i appear clearly to be the "second order" coordinates.

Further, consider the map $y^p = f^p(x^i)$ and write $y^p + \bar{u}^p t + \bar{v}^p \tau + \bar{w}^p t\tau$ for the p-th component of $(\mathbb{D} \otimes \mathbb{D})^n$. Since $t^2 = 0 = \tau^2$, (2.9) implies

$$y^p + \bar{u}^p t + \bar{v}^p \tau + \bar{w}^p t\tau = f^p + \frac{\partial f^p}{\partial x^i}(u^i t + v^i \tau + w^i t\tau) + \frac{\partial^2 f^p}{\partial x^i \partial x^j} u^i v^j t\tau \,.$$

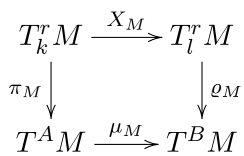
Passing to the individual components, we obtain the standard coordinate expression of TTf.

2.6 Natural transformations

Let $t \colon T^A \to T^B$ be a natural transformation of functors T^A and T^B. If we apply t to the addition and the multiplication of reals, we obtain two commutative diagrams of the form

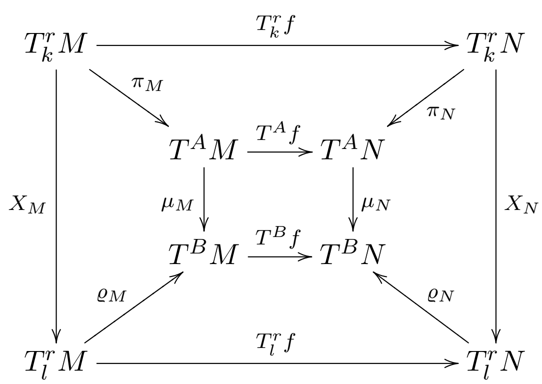
$$\begin{array}{ccc} A \times A & \longrightarrow & A \\ {\scriptstyle t_\mathbb{R}}\big\downarrow \;\; \big\downarrow{\scriptstyle t_\mathbb{R}} & & \big\downarrow{\scriptstyle t_\mathbb{R}} \\ B \times B & \longrightarrow & B \end{array} \tag{2.11}$$

This implies easily that $t_{\mathbb{R}}\colon A \to B$ is an algebra homomorphism.
The converse assertion is also true. Every algebra homomorphism $\mu\colon A \to B$ induces a natural transformation (denoted by the same symbol) $\mu\colon T^A \to T^B$ as follows. Every $X \in J^r_0(\mathbb{R}^l, \mathbb{R}^k)_0 = \mathrm{Hom}\,(\mathbb{D}^r_k, \mathbb{D}^r_l)$ defines a natural transformation $X_M\colon T^r_k M \to T^r_l M$ by the reparametrization $Y \to Y \circ X$, $Y \in T^r_k M$. For a general μ, we may consider the situation from Section 1.10 with $\sigma = X$. Taking into account that π_M and ϱ_M are surjective, we deduce by Section 1.10 that there is a unique map $\mu_M\colon T^A M \to T^B M$ making the following diagram commutative



(2.12)

Moreover, for every $f\colon M \to N$ the following diagram commutes



(2.13)

The inner square yields that μ_M form a natural transformation $T^A \to T^B$. Diagram (2.12) implies that there exists a polynomial map $\bar{\mu}\colon \mathbb{R}^l \to \mathbb{R}^k$ such that $\mu_M\colon T^A M \to T^B M$ is of the form

$$\mu_M(j^A\gamma) = j^B(\gamma \circ \bar{\mu}), \qquad \gamma\colon \mathbb{R}^k \to M. \tag{2.14}$$

Thus, we have proved

Proposition. *The natural transformations $T^A \to T^B$ are in bijection with the algebra homomorphisms $\mu\colon A \to B$. If $wA = k$ and $wB = l$, then there exists a polynomial map $\bar{\mu}\colon \mathbb{R}^l \to \mathbb{R}^k$ such that μ_M is of the form* (2.14).

In other words, even in the case of an arbitrary algebra homomorphism $\mu\colon A \to B$, the natural transformation μ_M is determined by a kind of

reparametrization. The admissibility of μ in the sense of Section 1.10 depends on $j^A\bar{\mu}$ only.
For example, the natural transformations $TT \to TT$, that correspond to the algebra homomorphisms $\mathbb{D}\otimes\mathbb{D} \to \mathbb{D}\otimes\mathbb{D}$ from Section 1.14, can be immediately expressed in this way. Each of them can be easily interpreted as a geometric construction on the iterated tangent bundle, [Ko84].

2.7 Multilinear maps

For every vector space V, the vector space structure on T^AV is defined by

$$j^A\gamma + j^A\delta = j^A(\gamma+\delta)\,, \quad cj^A\gamma = j^A(c\gamma)\,, \qquad c\in\mathbb{R} \tag{2.15}$$

with pointwise addition and scalar multiplication on the right-hand sides. Consider the map $\otimes\colon V\times A \to T^AV$

$$\otimes\big(v, j^A\varphi(t_1,\dots,t_k)\big) = j^A\big(\varphi(t_1,\dots,t_k)v\big)\,, \qquad v\in V,\ \varphi\colon\mathbb{R}^k\to\mathbb{R}\,. \tag{2.16}$$

In coordinates, we have $V=\mathbb{R}^m$, $T^AV = A^m$ and (2.16) is of the form

$$\big((v_1,\dots v_n),a\big)\mapsto(v_1a,\dots,v_na)\,,\quad v_i\in\mathbb{R},\ a\in A,\ i=1,\dots,n\,.$$

This implies $T^AV = V\otimes A$. In particular, if $A=\mathbb{D}$, then $V\otimes\mathbb{D} = V\times V$ and $TV = V\times V$ is the classical expression of the tangent bundle of a vector space.
Consider another vector space W and a linear map $f\colon V\to W$. Then

$$T^Af(v\otimes a) = T^Af\big(j^A(\varphi v)\big) = j^A\big(\varphi f(v)\big) = f(v)\otimes a\,.$$

One verifies directly that $T^Af\colon T^AV\to T^AW$ is also a linear map. Hence $T^Af = f\otimes \mathrm{id}_A$.

Proposition. *Let $f\colon V_1\times\dots\times V_l\to W$ be a multilinear map. Then $T^Af\colon V_1\otimes A\times\dots\times V_l\otimes A\to W\otimes A$ is also multilinear and*

$$T^Af(v_1\otimes a_1,\dots,v_l\otimes a_l) = f(v_1,\dots,v_l)\otimes a_1\dots a_l \tag{2.17}$$

where the product $a_1\dots a_l$ is in A.

Proof. The multilinearity of f implies

$$T^Af\big(j^A(\varphi_1v_1),\dots,j^A(\varphi_lv_l)\big) = j^A\big((\varphi_1\dots\varphi_l)f(v_1,\dots,v_l)\big)$$

with the pointwise product $\varphi_1(t_1,\dots,t_k)\dots\varphi_l(t_1,\dots,t_k)$ in $\mathbb{R}$. □

Corollary. *If (V,f) is an algebra, then $T^AV = V\otimes A$ is also an algebra with the multiplication determined by*

$$T^Af(v_1\otimes a_1, v_2\otimes a_2) = f(v_1,v_2)\otimes a_1a_2\,.$$

2.8 Natural transformations on vector spaces

For every vector space V and every algebra homomorphism $\mu\colon A \to B$,

$$\mu_V\colon T^A V = V \otimes A \to V \otimes B = T^B V$$

is defined by the reparametrization (2.14), so that μ_V is a linear map. Applying μ_V to (2.16), we obtain $\mu_V(v \otimes a) = v \otimes \mu(a)$. Hence we have

$$\mu_V = \mathrm{id}_V \otimes \mu\colon V \otimes A \to V \otimes B\,. \tag{2.18}$$

2.9 The iteration

By Section 2.4, the Weil functors coincide with the product preserving bundle functors on $\mathcal{M}f$. Since the iteration $T^A T^B$ of two Weil functors preserves products as well, this must be a Weil functor, too.

Proposition. *We have $T^A T^B = T^{B\otimes A}$.*

Proof. The Weil algebra of $T^A T^B$ is $(T^A T^B)(\mathbb{R}) = T^A(T^B \mathbb{R}) = T^A B = B \otimes A$. □

We know that every algebra homomorphism induces a natural transformation. In particular, the exchange isomorphism of the tensor product $ex\colon B \otimes A \to A \otimes B$ induces an exchange natural equivalence

$$ex_M\colon T^A T^B M \to T^B T^A M\,.$$

Geometrically, ex_M can be constructed as follows. Let $t \in \mathbb{R}^k$ and $\tau \in \mathbb{R}^l$. So every $Z \in T^A(T^B M)$ is of the form

$$Z = j^A\big(t \mapsto j^B(\tau \mapsto \delta(t,\tau))\big)$$

where $\delta\colon \mathbb{R}^k \times \mathbb{R}^l \to M$. Then

$$ex_M(Z) = j^B\big(\tau \mapsto j^A(t \mapsto \delta(t,\tau))\big)\,.$$

If we write $\pi^A_M\colon T^A M \to M$ for the bundle projection, then we have

$$T^A \pi^B_M = \pi^B_{T^A M} \circ ex_M\,, \quad T^B \pi^A_M \circ ex_M = \pi^A_{T^B M}\,. \tag{2.19}$$

In the special case $A = \mathbb{D}^r_k$, $B = \mathbb{D}^s_l$, which is important for applications, the natural exchange isomorphism $ex_M\colon T^r_k T^s_l M \to T^s_l T^r_k M$ can be expressed in the jet form

$$ex_M\big(j^r_0(t \mapsto j^s_0(\tau \mapsto \delta(t,\tau))\big) = j^s_0\big(\tau \mapsto j^r_0(t \mapsto \delta(t,\tau))\big)\,. \tag{2.20}$$

In the case $k = l = r = s = 1$, we evaluated all algebra homomorphisms $\mathbb{D} \otimes \mathbb{D} \to \mathbb{D} \otimes \mathbb{D}$ in Section 1.14. Clearly, the exchange isomorphism ex, that determines the well known canonical involution $TTM \to TTM$, corresponds to the values $c_2 = c_4 = 1$, $c_1 = c_3 = c_5 = c_6 = 0$.

2.10 The flow natural exchange

In the case of arbitrary A and $T^B = T$, we obtain a canonical exchange isomorphism

$$\varkappa_M^A \colon T^A TM \to TT^A M\,.$$

This map is said to be flow natural because of the following important property.

In general, let F be a bundle functor on the category $\mathcal{M}f_m$ of m-dimensional manifolds and local diffeomorphisms. The flow prolongation of a vector field $X \colon M \to TM$ is a vector field $\mathcal{F}X \colon FM \to T(FM)$ defined as follows. The flow Fl^X is locally a one-parameter family of diffeomorphisms $Fl_t^X \colon M \to M$, $t \in \mathbb{R}$. We construct $F(Fl_t^X) \colon FM \to FM$ for every t and we set

$$\mathcal{F}X = \frac{\partial}{\partial t}\Big|_0 F(Fl_t^X) \colon FM \to TFM\,.$$

In [KMS], it is deduced that the flow prolongation preserves the bracket of vector fields, i.e.

$$\mathcal{F}\big([X_1, X_2]\big) = [\mathcal{F}X_1, \mathcal{F}X_2] \tag{2.21}$$

for every two vector fields X_1, X_2 on M.

In the case of a Weil functor T^A, beside $\mathcal{T}^A X \colon T^A M = TT^A M$ we can construct the functorial prolongation $T^A X \colon T^A M \to T^A TM$ of the map X.

Proposition. *We have $\mathcal{T}^A X = \varkappa_M^A \circ T^A X$.*

Proof. Let $\bar{x} = \varphi(x,t)$ be the flow of X, $x \in M$, $t \in \mathbb{R}$. Hence $X(x) = \frac{\partial}{\partial t}\big|_0 \varphi(x,t)$. For $u = j^A\gamma(\tau) \in T^A M$, consider $\varphi(\gamma(\tau), t) \colon \mathbb{R}^k \times \mathbb{R} \to M$. Then

$$\frac{\partial}{\partial t}\Big|_0 j^A\big(\gamma(\tau), t\big) = \frac{\partial}{\partial t}\Big|_0 (T^A Fl_t^X)(u) = (\mathcal{T}^A X)(u)\,.$$

Of course, $\frac{\partial}{\partial t}\big|_0$ is identified with the first order jet with respect to t. If we exchange the order of j^A and $\frac{\partial}{\partial t}\big|_0$, we obtain

$$j^A\Big(\frac{\partial}{\partial t}\Big|_0 \varphi\big(\gamma(\tau), t\big)\Big) = j^A(X \circ \gamma) = \big(T^A X(u)\big)\,.$$

□

So the concrete evalutation of $\mathcal{T}^A X$ for a vector field $X = X^i(x^1, \dots, x^m)\frac{\partial}{\partial x_i}$ is also based on Section 2.5. For example, if $T^A = T$ is the tangent functor, then (2.10) implies directly

$$\mathcal{T}X = X^i \frac{\partial}{\partial x^i} + \Big(\frac{\partial X^i}{\partial x^j} x_1^j\Big) \frac{\partial}{\partial x_1^i}\,. \tag{2.22}$$

2.11 Fiber products

If F and G are two product preserving bundle functors on $\mathcal{M}f$, then the bundle functor $F \oplus G$ on $\mathcal{M}f$ defined by $(F \oplus G)(M) = FM \times_M GM$ and $(F \oplus G)(f) = Ff \times_f Gf$ also preserves products.

Proposition. *We have $T^A \oplus T^B = T^{A \oplus B}$.*

Proof. $T^A\mathbb{R}$ or $T^B\mathbb{R}$ is the product fibered manifold $\mathbb{R} \times N_A \to \mathbb{R}$ or $\mathbb{R} \times N_B \to \mathbb{R}$, respectively. Hence $(T^A \oplus T^B)(\mathbb{R}) = \mathbb{R} \times N_A \times N_B \to \mathbb{R}$. One verifies easily that the induced multiplication is that one from Section 1.12. □

2.12 Underlying functors

An interesting feature of the theory of Weil bundles is that every r-th order Weil functor T^A induces the underlying k-th order functors for all $k \le r = \operatorname{ord} A$.
Clearly, N_A^{k+1} is an ideal in A. Write $\pi_k \colon A \to A/N_A^{k+1}$ for the factor projection.

Definition. The factor algebra $A_k = A/N_A^{k+1}$ is called the underlying Weil algebra of order k. The Weil functor T^{A_k} is said to be the underlying k-th order functor of T^A.

So $(\pi_k)_M \colon T^A M \to T^{A_k} M$ is a surjective natural transformation. The following lemma is a direct consequence of $\mu(N_A) \subset N_B$.

Lemma. *For every algebra homomorphism $\mu \colon A \to B$, we have $\mu(N_A^k) \subset N_B^k$.*

Hence μ factorizes through an underlying algebra homomorphism $\mu_k \colon A_k \to B_k$. From the geometric point of view, every natural transformation $t \colon T^A \to T^B$ is projectable over a natural transformation $t_k \colon T^{A_k} \to T^{B_k}$ for every $k \le r$.
For example, the underlying first order functor of the iterated tangent functor TT is $T \oplus T$.

2.13 The affine structures

In the case of $k = r - 1$, there is a remarkable affine structure on $T^A M \to T^{A_{r-1}} M$. Consider a smooth map $\varphi\colon M \to Q$. Since N^r is a vector space, for every $Z \in T_x M \otimes N^r = \mathrm{Lin}(T_x^* M, N^r)$ and every $S \in T_x^* M$ we have $Z(S) \in N^r$, $x \in M$. We recall that every $X \in T_x^A M$ can be interpreted as an algebra homomorphism

$$Xf = f(x) + \widetilde{X} f\,, \quad \widetilde{X} f \in N,\ f \in C^\infty M\,. \tag{2.23}$$

Then, for every $Z \in T_x M \otimes N^r$,

$$(X + Z)f := f(x) + \tilde{X} f + Z(T_x^* f), \quad f \in C^\infty M \tag{2.24}$$

is also an algebra homomorphism. Indeed, for another $g \in C^\infty M$, we have

$$\begin{aligned} &\big(f(x) + \widetilde{X} f + Z(T_x^* f)\big)\big(g(x) + \widetilde{X} g + Z(T_x^* g)\big) \qquad (2.25)\\ &= \big(f(x) + \widetilde{X} f\big)\big(g(x) + \widetilde{X} g\big) + Z\big(f(x) T_x^* g + g(x) T_x^* f\big) = (X + Z)(fg)\,, \end{aligned}$$

as the other three terms vanish by virtue of $N N^r = 0$.
Clearly, X and $X + Z$ satisfy $(\pi_{r-1})_M(X) = (\pi_{r-1})_M(X + Z)$. Conversely, let X, $Y \in T_x^A M$ satisfy $(\pi_{r-1})_M(X) = (\pi_{r-1})_M(Y)$. Then

$$Yf = f(x) + \widetilde{X} f + Df \quad \text{with} \quad Df \in N^r\,. \tag{2.26}$$

Since X and Y are algebra homomorphisms, we have

$$D(fg) = \big(f(x) + \widetilde{X} f + Df\big)\big(g(x) + \widetilde{X} g + Dg\big) - X(fg) = f(x) Dg + g(x) Df$$

by virtue of $N N^r = 0$. So D is an N^r-valued derivation in $C^\infty M$ at x. In the classical way, see the proof in 1.5 of [KMS], we deduce $D \in T_x M \otimes N^r$. Hence

$$(\pi_{r-1})_M\colon T^A M \to T^{A_{r-1}} M \tag{2.27}$$

is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes N^r$ over $T^{A_{r-1}} M$.
Consider a smooth map $\varphi\colon M \to Q$.

Proposition. *$T^A \varphi\colon T^A M \to T^A Q$ is an affine bundle morphism over $T^{A_{r-1}} \varphi\colon T^{A_{r-1}} M \to T^{A_{r-1}} Q$, whose associated vector bundle morphism is the pullback of $T\varphi \otimes \mathrm{id}_{N^r}$.*

Proof. Let D be as in (2.26). For every $f \in C^\infty Q$, we write $(\varphi_x D)(f) = D(f \circ \varphi)$. Since D is an N^r-valued derivation at $x \in M$, $\varphi_x D$ is an N^r--valued derivation at $\varphi(x) \in Q$. □

Example. In the special case of T^r_k, we obtain the classical result that $T^r_k M \to T^{r-1}_k M$ is an affine bundle, whose associated vector bundle is the pullback of $TM \otimes S^r\mathbb{R}^{k*}$ over $T^{r-1}_k M$, [KMS]. More generally, consider the iterated velocities functor $T^s_l T^r_k$ of order $r+s$, whose Weil algebra is $\mathbb{D}^r_k \otimes \mathbb{D}^s_l$. One finds directly that the underlying bundle of the order $r+s-1$ is

$$T^{s-1}_l T^r_k M \times_{T^{s-1}_l T^{r-1}_k M} T^s_l T^{r-1}_k M$$

and the vector bundle in question is

$$TM \otimes (S^r\mathbb{R}^{k*}) \otimes (S^s\mathbb{R}^{l*})\,.$$

2.14 Regular A-velocities

Consider the vector space $V_A = N_A/N_A^2$. One finds directly that the underlying Weil algebra of the first order is $A_1 = \mathbb{R} \times V_A$ with the zero multiplication in V_A. This implies $T^{A_1}M = TM \otimes V_A$.

Definition. An A-velocity $X \in T^A_x M$ is called regular, if the linear map $V_A^* \to T_x M$ determined by $\pi_1(X) \in T_x M \otimes V_A$ is injective.

In the classical case of $X \in (T^r_k M)_x$, $\pi_1(X) \in T^1_k M$ is identified with a k-tuple of vectors in $T_x M$ and X is regular, if and only if these vectors are linearly independent. In general, Definition 2.1 implies that an A-velocity $j^A\gamma$ is regular, if and only if γ is an immersion at $0 \in \mathbb{R}^k$.

2.15 Contact A-elements

We recall that a contact (k,r)-element on a manifold M, as introduced by C. Ehresmann, is the set $Z \circ G^r_k$, where Z is a regular (k,r)-velocity on M and $G^r_k \subset J^r_0(\mathbb{R}^k, \mathbb{R}^k)_0$ is the submanifold of all invertible r-jets, [KMS]. The space of all such elements is a fiber bundle $K^r_k M$ over M.
This idea can be directly extended to A-velocities.

Definition. A contact A-element on a manifold M is the set $(\operatorname{Aut} A)(Z)$, where Z is a regular A-velocity on M.

All contact A-elements on M form a fiber bundle $K^A M \to M$.

2.16 Natural lifting of vector fields to $K^A M$

All results of this section are valid for an arbitrary Weil algebra. However, we have to point out that there are further geometric problems, in which certain special kinds of Weil algebras appear. We mention solely a paper by M. Kureš and W. Mikulski, [KuMi], on the natural operators transforming vector fields from a manifold M into vector fields on the bundle $K^A M$ of contact A-elements. In [KuMi], the best results are deduced for homogeneous Weil algebras. We recall that an ideal $I \subset \mathbb{R}[x_1, \dots, x_k]$ is said to be homogeneous, if $P \in I$ implies that all homogeneous components of polynomial P are also in I. A Weil algebra is called homogeneous, if it can be expressed in the form $\mathbb{R}[x_1, \dots, x_k]/I$ with a homogeneous ideal I. Clearly, all algebras $\mathbb{D}_k^r$ and their tensor products are homogeneous. Examples of nonhomogeneous Weil algebras are constructed in [KuMi].

Chapter 3

On the geometry of T^A-prolongations

3.1 Natural tensor fields of type $(1,1)$

Every $a \in A$ defines a natural tensor field $L(a)_M$ of type $(1,1)$ on T^AM for every manifold M as follows. The multiplication of the tangent vectors of M by reals is a map $\sigma_M\colon \mathbb{R}\times TM \to TM$. Applying T^A, we obtain $T^A\sigma_M\colon A\times T^ATM \to T^ATM$. Then we construct

$$\mathcal{T}^A\sigma_M := (\varkappa^A_M)^{-1}\circ T^A\sigma_M\circ(\mathrm{id}_A\times\varkappa^A_M)\colon A\times TT^AM \to TT^AM \tag{3.1}$$

and define $L(a)_M = \mathcal{T}^A\sigma_M(a,-)\colon TT^AM\to TT^AM$.
Since the multiplication in A is induced by the multiplication of reals, we have

$$L(a_1)_M\circ L(a_2)_M = L(a_1a_2)_M\,. \tag{3.2}$$

(In coordinates, (3.2) is a direct consequence of formula (3.3) below.) Clearly, $L(1)_M = \mathrm{id}_{TT^AM}$. The naturality of $L(a)$ means $TT^Af\circ L(a)_M = L(a)_N\circ TT^Af$ for every map $f\colon M\to N$.
To find the coordinate expression of $L(a)$, we take $M=\mathbb{R}^m$. Then $TT^A\mathbb{R}^m = A^m\times A^m$ and our definition implies directly

$$L(a)_{\mathbb{R}^m}(b_1,\dots,b_m,c_1,\dots,c_m) = (b_1,\dots,b_m,ac_1,\dots,ac_m),\quad a,\ b_i,\ c_i\in A\,. \tag{3.3}$$

In the case of the tangent functor, i.e. $A=\mathbb{D}=\{a+be\}$, the map $L(e)_M\colon TTM\to VTM$ is frequently used in analytical mechanics.

3.2 Natural vector fields

Every element D of the Lie algebra $\mathfrak{Aut}A = \operatorname{Der} A$ is of the form

$$D = \frac{d}{dt}\Big|_0 \gamma\,, \quad \gamma\colon \mathbb{R} \to \operatorname{Aut} A\,.$$

The natural transformations $\gamma_M(t)\colon T^A M \to T^A M$ determine a vertical vector field D_M on $T^A M$

$$D_M(y) = \frac{\partial}{\partial t}\Big|_0 \gamma(t)_M(y)\,, \quad y \in T^A M\,. \tag{3.4}$$

For example, in the case $A = \mathbb{D}$ we obtain the classical Liouville vector field on TM and its constant multiplies.

Remark. We find worth mentioning one of the oldest geometric results deduced by the technique of Weil algebras. The problem was to determine all natural operators transforming every vector field X on a manifold M into a vector field on $T^A M$. In [Ko88], see also [KMS], it is proved that every such operator is of the form

$$X \mapsto L(a)_M \circ \mathcal{T}^A X + D_M$$

for all $a \in A$ and $D \in \operatorname{Der} A$, $\mathcal{T}^A X$ being the flow prolongation of X. This result was new even in the case of the classical (k,r)-velocities, i.e. $T^A = T^r_k$.

3.3 T^A-prolongation of functions and vector fields

Every function $f\colon M \to \mathbb{R}$ induces a vector valued function $T^A f\colon T^A M \to A$. Every vector field Y on $T^A M$ determines the Lie derivative $Y(T^A f)\colon T^A M \to A$ of such vector valued function. Given $a \in A$, we define $aT^A f\colon T^A M \to A$ by multiplying in A.

Lemma. *If two vector fields Y and $\widetilde{Y}$ on $T^A M$ satisfy $Y(T^A f) = \widetilde{Y}(T^A f)$ for all $f\colon M \to \mathbb{R}$, then $Y = \widetilde{Y}$.*

Proof. By locality, it suffices to discuss the case $M = \mathbb{R}^n$, so that $T^A M = A^n$. Consider $f_i\colon \mathbb{R}^n \to \mathbb{R}$, $f_i(x^1, \dots x^n) = x^i$. Then $T^A f_i\colon A^n \to A$, $T^A f_i(a^1, \dots, a^n) = a^i$. Write $Y = \sum_{i=1}^n Y^i \frac{\partial}{\partial a^i}$, $\overline{Y} = \sum_{i=1}^n \overline{Y}^i \frac{\partial}{\partial a^i}$. Hence $Y(T^A f_i) = \overline{Y}(T^A f_i)$ implies $Y^i = \overline{Y}^i$ for every i. □

Further, for every vector field X on M and every $f\colon M \to \mathbb{R}$,

$$T^A(Xf) = \mathcal{T}^A X(T^A f)\,. \tag{3.5}$$

By locality, it suffices to dicuss the case $M = \mathbb{R}^m$. Let $X^i(x^1, \dots, x^m)$ or $f(x^1, \dots, x^m)$ be the coordinate expression of X or f, respectively. Hence $Xf = \sum\limits_{i=1}^{m} X^i \frac{\partial f}{\partial x^i}$ and the left hand side of (3.5) is

$$\sum_{i=1}^{m} (T^A X^i) T^A\Big(\frac{\partial f}{\partial x^i}\Big)\,. \tag{3.6}$$

By Section 2.10, the coordinate expression of $\mathcal{T}^A X$ is $T^A X^i$. Further, $\frac{\partial}{\partial x^i}(T^A f) = T^A\big(\frac{\partial f}{\partial x^i}\big)$. Hence the right hand side of (3.5) also coincides with (3.6).
Next, for every $X \in C^\infty TM$, every $f\colon M \to \mathbb{R}$ and every $a \in A$, it holds

$$\mathcal{T}^A X(aT^A f) = aT^A(Xf) \quad \text{and} \quad \big(L(a)\mathcal{T}^A X\big)T^A f = aT^A(Xf)\,. \tag{3.7}$$

In fact, we have $X(tf) = t(Xf)$ for all $t \in \mathbb{R}$. By (3.5), we obtain $\mathcal{T}^A X(aT^A f) = aT^A(Xf)$. Further, we have $(tX)f = t(Xf)$ for all $t \in \mathbb{R}$. Using (3.5) and the definition of $L(a)$, we obtain $(L(a)\mathcal{T}^A X)T^A f = aT^A(Xf)$.
Finally, for every vector fields X_1, X_2 on M and every a_1, $a_2 \in A$, we have

$$\big[L(a_1)\mathcal{T}^A X_1, L(a_2)\mathcal{T}^A X_2\big] = L(a_1 a_2)\mathcal{T}^A\big([X_1, X_2]\big)\,. \tag{3.8}$$

Indeed, by (2.21) the flow prolongation $\mathcal{T}^A$ preserves the bracket of vector fields. For every vector fields Y_1, Y_2 on $T^A M$ and every $F\colon T^A M \to A$, we have $[Y_1, Y_2]F = Y_1(Y_2 F) - Y_2(Y_1 F)$ by definition. Using (3.5) and (3.7), we obtain

$$\begin{aligned} \big[L(a_1)\mathcal{T}^A X_1, L(a_2)\mathcal{T}^A X_2\big](T^A f) &= L(a_1)\mathcal{T}^A X_1\big(a_2 T^A(Xf)\big) \\ -L(a_2)\mathcal{T}^A X_2\big(a_1 T^A(X_1 f)\big) &= a_1 a_2\big(T^A(X_1 X_2 f) - T^A(X_2 X_1 f)\big) \\ = a_1 a_2 \mathcal{T}^A[X_1, X_2]T^A f &= L(a_1 a_2)\mathcal{T}^A\big([X_1, X_2]\big)(T^A f)\,. \end{aligned}$$

Then our assertion follows from the above Lemma.

3.4 Semidirect products of Lie groups

An important phenomenon is that the prolongation $T^A G$ of every Lie group G has a canonical structure of semidirect product. So we start with a short survey of the general theory of semidirect products of Lie groups, [Bo].

Let H and K be Lie groups and $\varphi\colon H \to \operatorname{Aut} K$ be a group homomorphism of H into the group $\operatorname{Aut} K$ of all automorphisms of K. Then we define a group structure $H \rtimes_\varphi K$ on $H \times K$ by

$$(h_1, k_1)(h_2, k_2) = \big(h_1 h_2, \varphi(h_2^{-1})(k_1)k_2\big)\,, \tag{3.9}$$

h_1, $h_2 \in H$, k_1, $k_2 \in K$. Indeed, the associativity of (3.9) follows from

$$\begin{aligned}\big(h_1h_2, \varphi(h_2^{-1})(k_1)k_2\big)(h_3, k_3) &= \big(h_1h_2h_3, \varphi(h_3^{-1})\big(\varphi(h_2^{-1})(k_1)k_2\big)k_3\big)\\ &= (h_1h_2h_3, \varphi((h_2h_3)^{-1})(k_1)\varphi(h_3^{-1})(k_2)k_3\big)\end{aligned} \tag{3.10}$$

and the inverse element to (h,k) is $(h^{-1}, \varphi(h)(k^{-1}))$.
We say that (3.9) is a semidirect product of H and K. For short, we sometimes write $H \rtimes K$ only.
Further, if G, H and K are Lie groups, $p\colon G \to H$ is a group epimorphism, $K \subset G$ is the kernel of p and $i\colon K \to G$ is the group injection, then every group homomorphism $q\colon H \to G$ splitting p

$$\{e\} \longrightarrow K \xrightarrow{\ i\ } G \underset{p}{\overset{q}{\rightleftarrows}} H \longrightarrow \{e\} \tag{3.11}$$

defines a decomposition

$$G = H \times K\,, \quad g \mapsto \big(p(g), q(p(g))\big)\,,$$

where $q(p(g)) \in K$ by the exactness of (3.11). Then the rule

$$\varphi(h)(k) = q(h)\big(i(k)\big) \tag{3.12}$$

defines a group semidirect product structure $H \rtimes_q K$ on G.

3.5 T^A-prolongation of Lie groups

Consider the Weil bundle T^AG of a Lie group G. Then

$$\pi^A_G\colon T^AG \to G \tag{3.13}$$

is a group epimorphism with $\operatorname{Ker} \pi^A_G = T^A_eG$, where e is the unit of G, and the map

$$\nu^A_G\colon G \to T^AG \tag{3.14}$$

from (2.6) is a group homomorphism splitting (3.13).

Since the natural exchange $\varkappa^A_G\colon TT^AG \to T^ATG$ is constructed by means of flows, it maps T^A_eG into $T^A\mathfrak{g} = \mathfrak{g}\otimes A$. Hence

$$\mathrm{Lie}\,(T^AG) = T^A\mathfrak{g} = \mathfrak{g}\otimes A\,. \tag{3.15}$$

We have action Ad of G on $\mathfrak{g}\otimes A$ linearly generated by

$$\mathrm{Ad}(g)(x\otimes a) = \mathrm{Ad}(g)(x)\otimes a\,,\quad x\in\mathfrak{g}, g\in G, a\in A\,, \tag{3.16}$$

which corresponds to the splitting in (3.11).
By Section 3.4, T^AG has the structure of semidirect product

$$T^AG = G\rtimes(\mathfrak{g}\otimes A)\,. \tag{3.17}$$

In particular, the bracket in $T^A\mathfrak{g}$ is the T^A-prolongation of the bracket in $\mathfrak{g}$. By Section 2.7, we obtain

$$[v_1\otimes a_1, v_2\otimes a_2]_{\mathfrak{g}\otimes A} = [v_1,v_2]_{\mathfrak{g}}\otimes a_1a_2\,. \tag{3.18}$$

Remark. Formula (3.18) is powerful even in the case of the classical (k,r)-velocities, where $\mathrm{Lie}\,(T^r_kG) = \mathfrak{g}\otimes\mathbb{D}^r_k$. For example, in the case of the tangent functor T we obtain the well known expression of the bracket of $\mathrm{Lie}\,(TG) = \mathfrak{g}\times\mathfrak{g}$

$$\big[(v_1,\bar v_1),(v_2,\bar v_2)\big] = \big([v_1,v_2],[v_1,\bar v_2]+[\bar v_1,v_2]\big)\,. \tag{3.19}$$

Moreover, since the exponential map $\exp_G\colon \mathfrak{g}\to G$ is defined by means of flow, the exponential map of T^AG satisfies

$$\exp_{T^AG} = T^A(\exp_G)\colon T^A\mathfrak{g}\to T^AG\,. \tag{3.20}$$

3.6 T^A-prolongation of actions

Let $l\colon G\times M\to M$ be a left action of G on a manifold M. One verifies directly that $T^Al\colon T^AG\times T^AM = T^AM$ is a left action of T^AG on T^AM. For every algebra homomorphism $\mu\colon A\to B$, the natural transformations $\mu_G\colon T^AG\to T^BG$ and $\mu_M\colon T^AM\to T^BM$ form a morphism of actions.
The same is true for the right actions of G on M.
The infinitesimal action $\lambda\colon\mathfrak{g}\times M\to TM$ of l is defined by

$$\lambda := \mathrm{Inf}(l) = Tl\circ(i_G\times 0_M)\,, \tag{3.21}$$

where $i_G\colon \mathfrak{g} \to TG$ is the canonical injection and $0_M\colon M \to TM$ is the zero section. We write $\lambda(v) = \lambda(v, -)\colon M \to TM$ for the fundamental vector field on M determined by $v \in \mathfrak{g}$.
One finds easily that the infinitesimal action of $T^A\lambda$, which will be denoted by $\mathcal{T}^A\lambda\colon T^A\mathfrak{g} \times T^AM \to TT^AM$, is of the form $\mathcal{T}^A\lambda = \varkappa^A_M \circ T^A\lambda$.
Consider now the case $M = V$ is a vector space. Then $TV = V \times V$ and the first component of $\lambda\colon \mathfrak{g} \times V \to V \times V$ is the product projection $\mathfrak{g} \times V \to V$. The second component will be denoted by

$$\bar{\lambda}\colon \mathfrak{g} \times V \to V\,.$$

Since $T^AV = V \otimes A$ is also a vector space, we have $T^ATV = V \otimes A \times V \otimes A$. For the infinitesimal action $\mathcal{T}^A\lambda\colon T^A\mathfrak{g} \times T^AV \to TT^AV$, we obtain

$$\overline{\mathcal{T}^A\lambda}\colon T^A\mathfrak{g} \times T^AV \to T^AV\,. \tag{3.22}$$

Then our previous results yield

Corollary. *We have*

$$\overline{\mathcal{T}^A\lambda} = T^A\bar{\lambda}\colon T^A\mathfrak{g} \times T^AV \to T^AV\,.$$

In particular, let l be a linear action of G on V, so that $\bar{\lambda}$ is the classical representation of Lie algebra $\mathfrak{g}$ on V. Hence $\bar{\lambda}$ is a linear map. By Section 2.7, $\overline{\mathcal{T}^A\lambda}$ is of the form

$$\overline{\mathcal{T}^A\lambda}(v \otimes a_1, z \otimes a_2) = \bar{\lambda}(v, z) \otimes a_1a_2\,, \tag{3.23}$$

$v \in \mathfrak{g}$, $z \in V$, the product a_1a_2 being in A.

3.7 Vector bundles

For a vector bundle $p\colon E \to M$, $T^Ap\colon T^AE \to T^AM$ is also a vector bundle. Indeed, if $X_1, X_2 \in T^AE$ satisfy $T^Ap(X_1) = T^Ap(X_2)$, we may write $X_1 = j^A\varphi_1$, $X_2 = j^A\varphi_2$ with $p \circ \varphi_1 = p \circ \varphi_2$, so that $\varphi_1(u)$ and $\varphi_2(u)$ are in the same fiber of $E \to M$ for all $u \in \mathbb{R}^k$. Then we define $X_1 + X_2$ by $j^A(\varphi_1(u) + \varphi_2(u))$. Similarly, $c(j^A\varphi(u)) = j^A(c\varphi(u))$, $c \in \mathbb{R}$.
Further, if $\bar{p}\colon \bar{E} \to \bar{M}$ is another vector bundle and $f\colon E \to \bar{E}$ is a linear morphism over $\underline{f}\colon M \to \bar{M}$, then we deduce in the same way that $T^Af\colon T^AE \to T^A\bar{E}$ is a linear morphism over $T^A\underline{f}\colon T^AM \to T^A\bar{M}$.

3.8 Principal and associated bundles

Let $P(M,G)$ be a principal bundle with structure group G and projection $p\colon P \to M$. Write $\varrho_P\colon P \times G \to P$ for the right action of G on P. Then $T^A p\colon T^A P \to T^A M$ is a principle bundle with structure group $T^A G$ and $\varrho_{T^A P} = T^A \varrho_P\colon T^A P \times T^A G \to T^A P$.
Consider a fiber bundle $E = P[S,l]$ associated to P with respect to an action $l\colon G \times S \to S$ of G on the standard fiber S and write $q\colon E \to M$ for the bundle projection. Analogously to Section 3.7 we verify that $T^A q\colon T^A E \to T^A M$ is an associated bundle $T^A E = T^A P[T^A S, T^A l]$.
For every fibered manifold $Y \to M$, the vertical Weil bundle $V^A Y$ is the union

$$V^A Y = \bigcup_{x \in M} T^A(Y_x)\,, \quad V^A Y \subset T^A Y \tag{3.24}$$

of the Weil bundles of the individual fibers of Y. Clearly, $V^A P$ is a principal bundle $V^A P(M, T^A G)$ and $V^A E$ is an associated bundle $V^A E = V^A P[T^A S, T^A l]$.
The prolongation of the morphisms of principal and associated bundles will be discussed in Section 6.5.

3.9 Some further structures

The first paper dealing systematically with T^A-prolongations of various geometric structures is by A. Morimoto, [Mor]. In particular, he studied T^A-prolongation of almost complex structures and the prolongation of a classical linear connection on M into a classical linear connection on $T^A M$. Even the paper [GMP] by J. Gancarzewicz, W. Mikulski and Z. Pogoda is devoted, beside the general theory, to T^A-prolongation of further geometric structures. Main attention is paid to Riemannian and pseudo-Riemannian metrics, symplectic and almost symplectic structures, almost tangent structures and Kählerian structures.

3.10 Tensor fields of type $(1,k)$

A tensor field C of type $(1,k)$ on a manifold M can be considered as a map

$$C\colon TM \underbrace{\times_M \quad \times_M}_{k\text{-times}} TM \to TM\,.$$

Applying functor T^A, we obtain

$$T^A C\colon T^A TM \times_{T^A M} \cdots \times_{T^A M} T^A TM \to T^A TM\,.$$

Using the classical exchange $\varkappa_M^A$, we construct

$$\mathcal{T}^A C = \varkappa_M^A \circ T^A C \circ \big((\varkappa_M^A)^{-1} \times \cdots \times (\varkappa_M^A)^{-1}\big)\,. \tag{3.25}$$

This is a tensor field of type $(1,k)$ on $T^A M$, which is called the complete lift of C to $T^A M$. In the special case $k=0$, C is a vector field on M and $\mathcal{T}^A C$ is its flow prolongation, see Section 3.3.

Proposition. *For every $a_1, \dots, a_k \in A$,*

$$\mathcal{T}^A C\big(L(a_1)\mathcal{T}^A X_1, \dots, L(a_k)\mathcal{T}^A X_k\big) = L(a_1 \dots a_k)\mathcal{T}^A\big(C(X_1, \dots, X_k)\big)\,. \tag{3.26}$$

Proof. We have $C(t_1 X_1, \dots, t_k X_k) = t_1 \dots t_k C(X_1, \dots, X_k)$ for all $t_1, \dots, t_k \in \mathbb{R}$. Applying T^A to this relation and using the definition of $L(a)$, we obtain (3.26). □

3.11 Frölicher-Nijenhuis bracket

An antisymmetric tensor field P of type $(1,k)$ on M is said to be a tangent valued k-form on M. The Frölicher-Nijenhuis bracket is an important geometric operation on the tangent valued forms, see e.g. [KMS]. If Q is another tangent valued l-form on M, then the Frölicher-Nijenhuis bracket $[P,Q]$ is a tangent valued $(k+l)$-form on M defined by the formula

$$\begin{aligned}
&[P,Q](X_1, \dots, X_{k+l}) = \\
&\quad \frac{1}{k!l!}\sum_\sigma \bar\sigma\big[P(X_{\sigma_1}, \dots, X_{\sigma_k}), Q(X_{\sigma_{k+1}}, \dots, X_{\sigma_{k+l}})\big] \\
&\quad + \frac{-1}{k!(l-1)!}\sum_\sigma \bar\sigma Q\big(\big[P(X_{\sigma_1}, \dots, X_{\sigma_k}), X_{\sigma_{(k+1)}}\big], X_{\sigma_{(k+2)}} \dots\big) \\
&\quad + \frac{(-1)^{kl}}{(k-1)!l!}\sum_\sigma \bar\sigma P\big(\big[Q(X_{\sigma_1}, \dots, X_{\sigma_l}), X_{\sigma_{(l+1)}}\big], X_{\sigma_{(l+2)}} \dots\big) \\
&\quad + \frac{(-1)^{k-1}}{(k-1)!(l-1)!2}\sum_\sigma \bar\sigma Q\big(P\big([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \dots\big), X_{\sigma_{(k+2)}} \dots\big) \\
&\quad + \frac{(-1)^{(k-1)l}}{(k-1)!(l-1)!2}\sum \bar\sigma P\big(Q([X_{\sigma_1}, X_{\sigma_2}], X_{\sigma_3}, \dots), X_{\sigma_{(l+2)}} \dots\big)
\end{aligned} \tag{3.27}$$

where $X_1, \dots X_{k+l}$ are vector fields on M, the brackets on the right-hand side are the classical Lie bracket of vector fields, the summations are with respect to all permutations σ of $k+l$ letters and $\bar{\sigma}$ denotes the sign of the permutation σ. The tangent valued 0-forms are the vector fields and (3.27) reduces to the classical Lie bracket in the case $k = l = 0$.
The identity of TM is a special tangent valued 1-form on M and one verifies easily

$$\big[\operatorname{id}_{TM}, P\big] = 0 \tag{3.28}$$

for every tangent valued form P. By [KMS],

$$[P, Q] = -(-1)^{kl}[Q, P] \tag{3.29}$$

and the graded Jacobi identity holds

$$\big[P_1, [P_2, P_3]\big] = \big[[P_1, P_2], P_3\big] + (-1)^{k_1 k_2}\big[P_2, [P_2, P_3]\big] \tag{3.30}$$

for tangent valued k_i-forms P_i, $i = 1, 2, 3$.
Consider P and Q as above and express $[P, Q]$ in the form (3.27). Applying $\mathcal{T}^A$ to this expression and using Section 3.3, we obtain

Proposition. *We have*

$$\big[L(a_1)\mathcal{T}^A P, L(a_2)\mathcal{T}^A Q\big] = L(a_1 a_2)\mathcal{T}^A([P, Q])\,. \tag{3.31}$$

In particular, for $a_1 = a_2 = 1$ we obtain that the T^A-prolongation of tangent valued forms commutes with the Frölicher-Nijenhuis bracket.
The Frölicher-Nijenhuis bracket plays an important role in the theory of connections on arbitrary fibered manifolds, see Section 3.14 below.

3.12 Vector valued forms

Let V be a vector space. A V-valued k-form ω on M can be interpreted as a map

$$\omega \colon TM \times_M \cdots \times_M TM \to V\,. \tag{3.32}$$

Applying T^A and using $\varkappa^A_M$, we obtain

$$\mathcal{T}^A\omega \colon TT^A M \times_{T^A M} \cdots \times_{T^A M} TT^A M \to T^A V\,, \tag{3.33}$$

which is a $V \otimes A$-valued k-form on $T^A M$. Taking into account Section 3.3, we find easily that this operation commutes with the exterior differentiation, i.e.

$$\mathcal{T}^A(d\omega) = d(\mathcal{T}^A\omega)\,. \tag{3.34}$$

3.13 An infinitesimal-like algorithm

We describe the coordinate form of $\mathcal{T}^A\omega$. Let $M = \mathbb{R}^m$, $V = \mathbb{R}^n$ and

$$y^p = f^p_{i_1\dots i_k}(x^1,\dots,x^m)\,dx^{i_1}\wedge\cdots\wedge dx^{i_k} \tag{3.35}$$

be the coordinate expression of ω. We have $TT^A\mathbb{R}^m = A^m \times A^m$ and we write $a^i_1 = da^i$ for the corresponding algebra coordinates. The coordinate formula for $T^A f^p_{i_1\dots i_k}$ is described in Section 2.5. Then Section 2.10 implies that $\mathcal{T}^A\omega$ is of the form

$$\big(T^A f^p_{i_1\dots i_k}\big)\,da^{i_1}\wedge\cdots\wedge da^{i_k} \tag{3.36}$$

with all products in A.
For example, consider the tangent functor T and $n = 1$, $k = 2$, so that

$$\omega = f_{i_1 i_2}(x^1,\dots x^m)\,dx^{i_1}\wedge dx^{i_2}\,.$$

Using Section 2.10, we obtain

$$\begin{aligned}\mathcal{T}\omega &= \Big(f_{i_1i_2} + e\frac{\partial f_{i_1i_2}}{\partial x^i}\,x^i_1\Big)(dx^{i_1} + edx^{i_1}_1)\wedge(dx^{i_2}+edx^{i_2}_1) \\ &= f_{i_1i_2}dx^{i_1}\wedge dx^{i_2} \\ &\quad + e\Big(\Big(\frac{\partial f_{i_1i_2}}{\partial x^i}\,x^i_1\Big)\,dx^{i_1}\wedge dx^{i_2} + f_{i_1i_2}\big(dx^{i_1}\wedge dx^{i_2}_1 + dx^{i_1}_1\wedge dx^{i_2}\big)\Big)\,.\end{aligned} \tag{3.37}$$

Remark. It should be pointed out that this “infinitesimal-like” algorithm is similar to some ideas from the synthetic differential geometry, see [Kock] or [MoRe].
Further, since a tangent valued k-form P looks, in coordinates, like a vector valued k-form, the procedure of finding the coordinate expression of $\mathcal{T}^A P$ is quite analogous to (3.36).

3.14 T^A-prolongation of general connections

Consider an arbitrary fibered manifold $p\colon Y\to M$ and construct the bundle $TY \to Y\times_M TM$.

Definition. A general connection on $p\colon Y\to M$ is a section $\Gamma\colon Y\times_M TM\to TY$ linear in TM.
The coordinate form of Γ is

$$dy^p = F^p_i(x,y)\,dx^i \tag{3.38}$$

with arbitrary $F_i^p(x,y)$. In particular, if Y is a principal G-bundle $Y(M,G)$ and Γ is G-invariant, then Γ is the classical principal connection on $Y(M,G)$.

The projection $T_yY \to V_yY$ in the direction of $\Gamma(y)$ defines the connection form $\omega_\Gamma\colon TY \to VY \subset TY$. The absolute differentiation with respect to Γ is a map

$$\nabla_\Gamma\colon J^1Y \to VY \otimes T^*M \tag{3.39}$$

defined by $\nabla_\Gamma(j^1_x s) = j^1_x s - \Gamma(x, s(x))$, $x \in M$, where $J^1_{s(x)}Y$ is considered as an affine space associated to $V_{s(x)}Y \otimes T^*_x M$.
The curvature of Γ is a map $C\Gamma\colon Y \times_M \Lambda^2 TM \to VY$ that can be characterized as the obstruction for lifting the bracket

$$(C\Gamma)(y, X_1, X_2) = \big[\Gamma(X_1), \Gamma(X_2)\big](y) - \Gamma\big([X_1, X_2]\big)(y)\,, \tag{3.40}$$

where $y \in Y$ and X_1, X_2 are two vector fields on M. By direct evaluation we find that the coordinate form of (3.40) is

$$2\Big(\frac{\partial F_i^p}{\partial x^j} + \frac{\partial F_i^p}{\partial y^q} F_j^q\Big)\frac{\partial}{\partial y^p} \otimes dx^i \wedge dx^j\,. \tag{3.41}$$

Hence (3.40) can be interpreted as a tangent valued 2-form on Y.
Consider a Weil functor T^A and the flow natural exchange $\varkappa^A_M\colon T^ATM \to TT^AM$. Then we can define

$$\begin{aligned} \mathcal{T}^A\Gamma\colon (T^AY \times_{T^AM} TT^AM) \to TT^AY\,,\\ \mathcal{T}^A\Gamma(\eta,\xi) = \varkappa^A_Y\big(T^A\Gamma(\eta, \varkappa^A_M(\xi))\big)\,, \quad \eta \in T^AY,\ \xi \in TT^AM\,. \end{aligned} \tag{3.42}$$

This is linear in TT^AM as well, so we obtain a general connection $\mathcal{T}^A\Gamma$ on T^AM called the T^A-prolongation of Γ. Hence the coordinate expression of $\mathcal{T}^A\Gamma$ is

$$T^A dy^p = T^A\big(F_i^p(x,y)\big) T^A\, dx^i\,. \tag{3.43}$$

Chapter 4

Product preserving bundle functors on $\mathcal{FM}$

4.1 Local categories over manifolds

We do not find necessary to discuss here the concepts of local category $\mathcal{C}$ over manifolds and of bundle functor on $\mathcal{C}$ in full generality, [Eh], [Eil], and we refer the reader to Section 18 of the book [KMS] for more details. We recall that $\mathcal{C}$ is a category endowed with a faithful functor $m\colon \mathcal{C} \to \mathcal{M}f$. The assumption that the functor m is faithful means that every induced map $m_{A,B}\colon \mathcal{C}(A,B) \to C^\infty(mA, mB)$ is injective. The manifold mS is called the underlying manifold of an $\mathcal{C}$-object S and S is said to be $\mathcal{C}$-object over mS.
In this book, we shall need mainly the following cases:

$\mathcal{M}f$ – all smooth manifolds and all smooth maps,

$\mathcal{M}f_m \subset \mathcal{M}f$ – m-dimensional manifolds and local diffeomorphisms,

$\mathcal{FM}$ – all fibered manifolds and their morphisms,

$\mathcal{FM}_m \subset \mathcal{FM}$ – fibered manifolds with m-dimensional bases and local diffeomorphisms as base maps,

$\mathcal{FM}_{m,n} \subset \mathcal{FM}_m$ – fibered manifolds with m-dimensional bases and n-dimensional fibers and local fiber isomorphisms.

Definition. A category $\mathcal{C}$ over manifolds is called local, if every $S \in \operatorname{Ob}\mathcal{C}$ and every open subset $U \subset mS$ determine a $\mathcal{C}$-object $L(S,U)$ of S over U, called the localization of S over U, such that $L(S,mS) = S$,

$L(L(S,U),V) = L(S,V)$ for every $S \in \operatorname{Ob}\mathcal{C}$ and every open subsets $V \subset U \subset mS$ and the conditions of the aggregation of objects and of morphisms are satisfied.

All the above mentioned categories are local in this sense.

4.2 Bundle functors

We denote by $B\colon \mathcal{FM} \to \mathcal{M}f$ the functor of constructing bases.

Definition. Given a local category $\mathcal{C}$ over manifolds, a bundle functor on $\mathcal{C}$ is a functor $F\colon \mathcal{C} \to \mathcal{FM}$ satisfying $B \circ F = m$ and the localization condition:
for every $\mathcal{C}$-object S and every inclusion of an open subset $i_U\colon U \hookrightarrow mS$, $F(L(S,U))$ is the restriction $p_S^{-1}(U)$ of the value of $p_S\colon FS \to mS$ over U and Fi_U is the inclusion $p_S^{-1}(U) \hookrightarrow FS$.

4.3 The functor T^μ

Let $\mu\colon A \to B$ be an algebra homomorphism. By Sections 2.4 and 2.6, μ induces two bundle functors T^A, T^B on $\mathcal{M}f$ and a natural transformation $\mu\colon T^A \to T^B$. For every fibered manifold $p\colon Y \to M$, we have $T^B p\colon T^B Y \to T^B M$. Using the map $\mu_M\colon T^A M \to T^B M$, we construct the induced bundle

$$T^\mu Y = \mu_M^*(T^B Y) \xrightarrow{T^\mu p} T^A M\,, \tag{4.1}$$

which may be also written as

$$T^\mu Y = T^A M \times_{T^B M} T^B Y\,. \tag{4.2}$$

Proposition. *T^μ is a bundle functor on $\mathcal{FM}$ that preserves products.*

Proof. By (4.1), $T^\mu Y = \mu_M^* T^B Y$ and $T^\mu \bar{Y} = \mu_{\bar{M}}^* T^B \bar{Y}$. Hence $T^\mu Y \times T^\mu \bar{Y} = (T^A M \times_{T^B M} T^B Y) \times (T^A \bar{M} \times_{T^B \bar{M}} T^B \bar{Y}) = (T^A M \times T^A \bar{M}) \times_{T^B M \times T^B \bar{M}} (T^B Y \times T^B \bar{Y}) = T^\mu(Y \times \bar{Y})$. □

It is useful to express T^μ in the case of two product fibered manifolds $M\times N$, $\bar{M} \times \bar{N}$ and a product morphism $f \times g\colon M \times N \to \bar{M} \times \bar{N}$, $f\colon M \to \bar{M}$, $g\colon N \to \bar{N}$. Then

$$\begin{aligned} T^\mu(M \times N) &= T^A M \times T^B N\,,\\ T^\mu(\bar{M} \times \bar{N}) &= T^A \bar{M} \times T^B \bar{N}\,,\\ T^\mu(f \times g) &= T^A f \times T^B g\,. \end{aligned} \tag{4.3}$$

4.4 The converse assertion

Proposition. *For every product preserving bundle functor F on $\mathcal{FM}$, there exists a unique algebra homomorphism $\mu\colon A \to B$ such that $F = T^\mu$.*

Proof. Let pt denote one element manifold and $pt_M\colon M \to pt$ the unique map. There are two canonical injections $i_1,\ i_2\colon \mathcal{M}f \to \mathcal{FM}$ defined by $i_1M = (\mathrm{id}_M\colon M \to M)$, $i_1f = (f, f)$, $i_2M = (pt_M\colon M \to pt)$, $i_2f = (f, \mathrm{id}_{pt})$ and a natural transformation $t\colon i_1 \to i_2$, $t_M = (\mathrm{id}_M, pt_M)\colon i_1M \to i_2M$. Applying F, we obtain two bundle functors $F \circ i_1$, $F \circ i_2$ on $\mathcal{M}f$ and a natural transformation $F \circ t\colon F \circ i_1 \to F \circ i_2$. Clearly, both $F \circ i_1$ and $F \circ i_2$ preserve products. By Sections 2.4 and 2.6, there exists an algebra homomorphism $\mu\colon A \to B$ such that $F \circ i_1 = T^A$, $F \circ i_2 = T^B$ and $F \circ t = \mu$. Consider a commutative diagram

$$\begin{array}{ccc} (p\colon Y \to M) & \xrightarrow{(1_Y, pt_M)} & (pt_Y\colon Y \to pt) \\ \big\downarrow{\scriptstyle (p,\mathrm{id}_M)} & & \big\downarrow{\scriptstyle i_2 p} \\ (\mathrm{id}_M\colon M \to M) & \xrightarrow{t_M} & (pt_M\colon M \to pt) \end{array} \tag{4.4}$$

One verifies easily that (4.4) is a pullback in $\mathcal{FM}$. We have assumed that F preserves products and has the localization property. But every fibered manifold is locally a product and the induced bundle of a product is a product, so that F preserves inducing of bundles. If we apply F to (4.4), we obtain a pullback diagram

$$\begin{array}{ccc} FY & \longrightarrow & T^BY \\ \big\downarrow & & \big\downarrow{\scriptstyle T^B p} \\ T^AM & \xrightarrow{\mu_M} & T^BM \end{array} \tag{4.5}$$

This proves our claim. □

4.5 The natural transformations

Consider two algebra homomorphisms $\mu\colon A \to B$ and $\nu\colon C \to D$ and a natural transformation $\varphi\colon T^\mu \to T^\nu$. By Propositions 38.13 and 38.17 in [KMS], every $\varphi_Y\colon T^\mu Y \to T^\nu Y$ is projectable

$$\begin{array}{ccc} T^\mu Y & \xrightarrow{\varphi_Y} & T^\nu Y \\ {\scriptstyle p^\mu}\big\downarrow & & \big\downarrow{\scriptstyle p^\nu} \\ T^AM & \xrightarrow{\psi_M} & T^CM \end{array} \tag{4.6}$$

where ψ is a natural transformation $\psi\colon T^A \to T^C$. The related diagram of algebra homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\psi} & C \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ B & \xrightarrow{\varphi} & D \end{array} \tag{4.7}$$

commutes.
Conversely, two algebra homomorphisms $\psi\colon A \to C$ and $\varphi\colon B \to D$ such that (4.7) commutes determine a natural transformation $T^\mu \to T^\nu$ by (4.6). Further, consider the iteration $T^\nu(T^\mu Y)$. By Section 2.9, this corresponds to the tensor product $\mu \otimes \nu\colon A \otimes C \to B \otimes D$.
We remark that the results of Sections 4.3–4.5 were first deduced by W. Mikulski, [Mi]. Our presentation is mostly based on [DoKo01].

4.6 Fiber $(k, l; r, s)$-velocities

For a fibered manifold $p\colon Y \to M$ and a manifold Q, we define an (r, s)-jet of $\gamma\colon Y \to Q$ at $y \in Y$, $s \geq r$, to be the pair

$$j_y^{r,s}\gamma = \left(j_y^r\gamma, j_y^s(\gamma \mid Y_{p(y)})\right). \tag{4.8}$$

We denote by $J^{r,s}(Y, Q)$ the space of all such (r, s)-jets.
Write $\mathbb{R}^{k,l}$ for the fibered manifold $\mathbb{R}^{k+l} \to \mathbb{R}^k$. The bundle

$$T_{k,l}^{r,s}Q = J_{0,0}^{r,s}(\mathbb{R}^{k,l}, Q)$$

will be called the bundle of $(k, l; r, s)$-velocities on Q. Every smooth map $f\colon Q \to \bar{Q}$ defines $T_{k,l}^{r,s}f\colon T_{k,l}^{r,s}Q \to T_{k,l}^{r,s}\bar{Q}$ by

$$T_{k,l}^{r,s}f(j_{0,0}^{r,s}\gamma) = j_{0,0}^{r,s}(f \circ \gamma).$$

Clearly, $T_{k,l}^{r,s}$ is a product preserving bundle functor on $\mathcal{M}f$. According to Section 2.6, its Weil algebra is

$$\mathbb{D}_{k,l}^{r,s} := T_{k,l}^{r,s}\mathbb{R}.$$

Let $\mathcal{E}(k + l)$ be the algebra of all germs of smooth functions on $\mathbb{R}^{k+l}$ at 0, $\mathfrak{m}(k + l)$ be its maximal ideal, which is generated by the germs of all variables $x_1, \dots, x_{k+l}$, and $\mathfrak{m}(k) \subset \mathfrak{m}(k + l)$ be the ideal generated by the germs of $x_1, \dots, x_k$. By the definition of $T_{k,l}^{r,s}$, the ideal generating $\mathbb{D}_{k,l}^{r,s}$ is

$$\left\langle \mathfrak{m}(k + l)^{s+1}, \mathfrak{m}(k)\mathfrak{m}(k + l)^r \right\rangle. \tag{4.9}$$

Some geometric operations from Chapter 3 can be directly extended to $(k,l;r,s)$-velocities. If $\varkappa\colon G\times G\to G$ is a Lie group, then $T^{r,s}_{k,l}G\to G$ is a Lie group with multiplication

$$T^{r,s}_{k,l}\varkappa\big(j^{r,s}_{k,l}\gamma, j^{r,s}_{k,l}\delta\big) = j^{r,s}_{k,l}\big(\varkappa\circ(\gamma,\delta)\big)\,.$$

A similar formula extends the actions, both left and right, of G on a manifold M into an induced action of $T^{r,s}_{k,l}G$ on $T^{r,s}_{k,l}M$. In particular, a principal bundle $P(M,G)$ is extended into a principal bundle $T^{r,s}_{k,l}P(T^{r,s}_{k,l}M, T^{r,s}_{k,l}G)$. In the same way, the associated bundle $P[S,l]$ induces an associated bundle $T^{r,s}_{k,l}P\big[T^{r,s}_{k,l}S, T^{r,s}_{k,l}l\big]$.

4.7 (q,s,r)-jets of fibered morphisms

For the morphisms of fibered manifolds, we introduce a useful modification of the concept of r-jet of smooth maps. Consider two fibered manifolds $p\colon Y\to M$ and $\bar p\colon \bar Y\to \bar M$.

Definition. We say that two $\mathcal{FM}$-morphisms f, $g\colon Y\to\bar Y$ with the base maps $\underline{f}$, $\underline{g}\colon M\to\bar M$ determine the same (q,s,r)-jet $j^{q,s,r}_y f = j^{q,s,r}_y g$ at $y\in Y$, $s\ge q\le r$, if

$$j^q_y f = j^q_y g\,,\quad j^s_y(f\mid Y_x) = j^s_y(g\mid Y_x)\,,\quad j^r_x\underline{f} = j^r_x\underline{g}\,,\qquad x=p(y)\,. \tag{4.10}$$

We write $J^{q,s,r}(Y,\bar Y)$ for the bundle of all (q,s,r)-jets of Y into $\bar Y$.

The composition of $\mathcal{FM}$-morphisms defines the composition of (q,s,r)-jets. The base maps induce a canonical projection $J^{q,s,r}(Y,\bar Y)\to J^r(M,\bar M)$, where $J^r(M,\bar M)$ denotes the classical bundle of r-jets of M into $\bar M$.
Consider the product fibered manifold $\mathbb{R}^{k,l} = (\mathbb{R}^k\times\mathbb{R}^l\to\mathbb{R}^k)$. We introduce the bundle of fiber (k,l)-velocities of order (q,s,r) on a fiber manifold Y by

$$T^{q,s,r}_{k,l}Y = J^{q,s,r}_{0,0}(\mathbb{R}^{k,l},Y)\,. \tag{4.11}$$

For every $\mathcal{FM}$-morphism $f\colon Y\to\bar Y$, $T^{q,s,r}_{k,l}f\colon T^{q,s,r}_{k,l}Y\to T^{q,s,r}_{k,l}\bar Y$ is defined by the jet composition. Clearly, $T^{q,s,r}_{k,l}$ is a product preserving bundle functor on $\mathcal{FM}$.
By the definition of $T^{q,s,r}_{k,l}$, the corresponding Weil algebra $\mathbb{D}^{q,s,r}_{k,l}$ is generated by the ideal

$$\big\langle \mathfrak{m}(k+l)^{s+1}, \mathfrak{m}(k)\mathfrak{m}(k+l)^q, \mathfrak{m}(k)^{r+1}\big\rangle\,,\qquad s\ge q\le r\,, \tag{4.12}$$

and $T^{q,s,r}_{k,l}$ is defined by the canonical homomorphism $\mathbb{D}^{q,s,r}_{k,l} \to \mathbb{D}^{q,s}_{k,l}$.
Let $m = \dim M$ and $m + n = \dim Y$. Then $P^{q,s,r}Y = \operatorname{inv} J^{q,s,r}_{m,n}(\mathbb{R}^{m,n}, Y)$ is a principal bundle over Y with structure group $G^{q,s,r}_{m,n} = \operatorname{inv} J^{q,s,r}_{0,0}(\mathbb{R}^{m,n}, \mathbb{R}^{m,n})_{0,0}$, which is called the (q, s, r)-th order frame bundle of Y. The associated bundle $P[S, l]$ induces an associated bundle $P^{q,s,r}Y[T^{q,s,r}_{m,n}S, T^{q,s,r}_{m,n}l]$ in a similar way to Section 4.6.

Chapter 5

Fiber product preserving bundle functors

5.1 The fundamental jet functors

There are 3 bundle functors, whose construction is based on the concept of r-jet only.

For every two manifolds M and N, $J^r(M,N)$ is the bundle of all r-jets of M into N. Let $f\colon M \to \bar{M}$ be a local diffeomorphism and $g\colon N \to \bar{N}$ be a map. Then the induced map $J^r(f,g)\colon J^r(M,N) \to J^r(\bar{M},\bar{N})$ is defined by

$$J^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \tag{5.1}$$

where x or y is the source or the target of $X \in J^r(M,N)$. Hence J^r is a bundle functor defined on the product category $\mathcal{M}f_m \times \mathcal{M}f$, $m = \dim M$.

For every fibered manifold $p\colon Y \to M$, J^rY is the bundle of r-jets of local sections of Y. If $\bar{p}\colon \bar{Y} \to \bar{M}$ is another fibered manifold and $f\colon Y \to \bar{Y}$ is an $\mathcal{FM}$-morphism such that the base map $\underline{f}\colon M \to \bar{M}$ is a local diffeomorphism, then the map $J^r(\underline{f}, f)\colon J^r(M,Y) \to J^r(\bar{M},\bar{Y})$ transforms J^rY into $J^r\bar{Y}$. The restricted and corestricted map $J^rf\colon J^rY \to J^r\bar{Y}$ is called the r-th jet prolongation of f. Hence J^r is a functor on the category $\mathcal{FM}_m$. In concrete problems, it is always clear which functor J^r is under consideration and we have no intention to change this convention. But in a theoretical research we have to distinguish. So in the theoretical part of this book we shall write J^r_h in the case of $\mathcal{FM}_m$.

The r-th vertical jet prolongation J^r_vY, which is used e.g. in some theories of higher order absolute differentiation, is defined by

$$J_v^r Y = \bigcup_{x \in M} J_x^r(M, Y_x)\,. \tag{5.2}$$

The restriction and corestriction of $J^r(\underline{f}, f)$ defines $J_v^r f\colon J_v^r Y \to J_v^r \bar{Y}$. Hence J_v^r is a bundle functor on $\mathcal{FM}_m$. In this context, we also say that $J_h^r Y$ is the r-th horizontal jet prolongation of Y.
The construction of product fibered manifolds is a functor

$$i\colon \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m\,, \quad i(M \times N) = (M \times N \to M) \tag{5.3}$$

and $i(f \times g)$ is $f \times g$ with the base map f. We have $J_h^r(M \times N) = J^r(M, N) = J_v^r(M \times N)$ and both functors $J_h^r \circ i$ and $J_v^r \circ i$ coincide with J^r.

5.2 Nonholonomic jets

The r-th nonholonomic prolongation $\widetilde{J}_h^r Y$ of a fibered manifold Y is defined by the iteration

$$\widetilde{J}_h^r Y = J_h^1(\widetilde{J}_h^{r-1} Y \to M)\,, \tag{5.4}$$

$\widetilde{J}_h^1 Y = J_h^1 Y$. For every $f\colon Y \to \bar{Y}$ in $\mathcal{FM}_m$, the iteration determines

$$\widetilde{J}_h^r f = J_h^1(\widetilde{J}_h^{r-1} f)\colon \widetilde{J}_h^r Y \to \widetilde{J}_h^r \bar{Y}\,. \tag{5.5}$$

Hence $\widetilde{J}_h^r$ is a bundle functor on $\mathcal{FM}_m$. The canonical inclusion $J_h^r Y \hookrightarrow \widetilde{J}_h^r Y$ is defined by the iteration

$$j_x^r s \mapsto j_x^1(u \mapsto j_u^{r-1} s) \tag{5.6}$$

for every local section s of Y, $u \in M$.
The restriction $\widetilde{J}_h^r \circ i$ yields a functor $\widetilde{J}^r$ on $\mathcal{M}f_m \times \mathcal{M}f$. The space

$$\widetilde{J}^r(M, N) = \widetilde{J}_h^r(M \times N \to M)$$

is the bundle of nonholonomic r-jets of manifold M into manifold N defined by Ehresmann, [Eh], p. 361. The elements of $J^r(M, N) \subset \widetilde{J}^r(M, N)$ are also said to be the holonomic r-jets.
Let Q be a third manifold. Ehresmann introduced the composition of nonholonomic r-jets by the following induction, [Eh], p. 361. For $r = 1$, we have the composition of 1-jets. Write $\beta\colon \widetilde{J}^{r-1}(M, N) \to N$ for the canonical

projection. Let $X = j^1_x s(u) \in \widetilde{J}^r_x(M,N)_y$, $u \in M$, and $Z = j^1_y\sigma \in \widetilde{J}^r_y(N,Q)$, $u = \beta(s(x))$. Then

$$Z \circ X := j^1_x\big(\sigma(\beta(s(u))) \circ s(u)\big) \in \widetilde{J}^r_x(M,Q) \tag{5.7}$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. If X and Z are holonomic r-jets, then (5.7) coincides with the classical composition.

The r-th vertical nonholonomic prolongation of Y is defined by

$$\widetilde{J}^r_v Y = \bigcup_{x\in M} \widetilde{J}^r_x(M, Y_x)\,.$$

Even this is a bundle functor on $\mathcal{FM}_m$. Similarly to the holonomic case, we have $\widetilde{J}^r_h \circ i = \widetilde{J}^r = \widetilde{J}^r_v \circ i$.

We remark that there are several geometrically important special classes of nonholonomic r-jets, the most remarkable ones are the semiholonomic r-jets. This subject will be discussed in more detail in Chapter 7.

5.3 Bundle functors in the product case

We start with some properties of bundle functors on $\mathcal{M}f \times \mathcal{M}f$ that are needed for the main subject of this chapter. First we introduce some notation.

Let F be a bundle functor on $\mathcal{M}f \times \mathcal{M}f$. We write $F_x(M,N)$ or $F(M,N)_y$ or $F_x(M,N)_y$ for the submanifold of all elements of $F(M,N)$ over $x \in M$ or $y \in N$ or $(x,y) \in M \times N$, respectively. For $g\colon M \to \bar{M}$ and $f\colon N \to \bar{N}$, we write $F_x(g,f)\colon F_x(M,N) \to F_{g(x)}(\bar{M},\bar{N})$ for the restricted and corestricted map.

Definition. We say that F preserves products in the second factor, if $F(M, N_1 \times N_2) = F(M,N_1) \times_M F(M,N_2)$. We say that F has order r in the first factor, if for every g, $\bar{g}\colon M \to \bar{M}$ and $f\colon N \to \bar{N}$, $j^r_x g = j^r_x \bar{g}$ implies

$$F_x(g,f) = F_x(\bar{g},f)\colon F_x(M,N) \to F_{g(x)}(\bar{M},\bar{N})\,.$$

Remark. Clearly, if F has order r in the standard sense, i.e. $j^r_{x,y}(g,f) = j^r_{x,y}(\bar{g},\bar{f})$ implies $F_x(g,f)_y = F_x(\bar{g},\bar{f})_y$, then F has order r in the first factor.

5.4 The case of $\mathcal{M}f_m \times \mathcal{M}f$

Let F be a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ that preserves products in the second factor. We define an associated bundle functor G^F on $\mathcal{M}f$ by $G^F(N) = F_0(\mathbb{R}^m, N)$ and $G^F(f) = F_0(\mathrm{id}_{\mathbb{R}^m}, f)\colon G^F N \to G^F \bar{N}$, $f\colon N \to \bar{N}$. Clearly, G^F preserves products, so that $G^F = T^A$ for a Weil algebra A. Assume further that F has order r in the first factor. For $X \in G^r_m$, $X = j^r_0\gamma$ and every manifold N, we set

$$H^F_N(X) = F_0(\gamma, \mathrm{id}_N)\colon G^F N \to G^F N\,.$$

Since the following diagram commutes

$$\begin{array}{ccc} (\mathbb{R}^m, N) & \xrightarrow{(\mathrm{id}_{\mathbb{R}^m}, f)} & (\mathbb{R}^m, \bar{N}) \\ {\scriptstyle(\gamma,\mathrm{id}_N)}\downarrow & & \downarrow{\scriptstyle(\gamma,\mathrm{id}_{\bar{N}})} \\ (\mathbb{R}^m, N) & \xrightarrow{(\mathrm{id}_{\mathbb{R}^m}, f)} & (\mathbb{R}^m, \bar{N}) \end{array} \tag{5.8}$$

each $H^F(X)$ is a natural equivalence $T^A \to T^A$. By Theorem 2.4, $H^F(X)$ corresponds to an element of $\mathrm{Aut}\,A$. Clearly, $H^F\colon G^r_m \to \mathrm{Aut}\,A$ is a group homomorphism.

Conversely, consider a Weil algebra A and a group homomorphism $H\colon G^r_m \to \mathrm{Aut}\,A$. For every manifold N, we have the induced left action H_N of G^r_m on $T^A N$, so that we can construct the associated bundle $P^r M[T^A N, H_N] =: (A, H)(M, N)$ for every m-dimensional manifold M. We underline that the elements of $P^r M[T^A N, H_N]$ are the equivalence classes $\{u, Z\}$, $u \in P^r M$, $Z \in T^A N$, with respect to the equivalence relation

$$\{u, Z\} \approx \{u \circ g, H_N(g^{-1})(Z)\}\,, \quad g \in G^r_m\,. \tag{5.9}$$

For every local diffeomorphism $g\colon M \to \bar{M}$, we have the induced morphism $P^r g\colon P^r M \to P^r \bar{M}$ of principal bundles and every map $f\colon N \to \bar{N}$ induces a G^r_m-equivariant map $T^A f\colon T^A N \to T^A \bar{N}$. So we can construct the morphism of associated bundles $(A, H)(g, f) := P^r g[T^A f]\colon P^r M[T^A N, H_N] \to P^r \bar{M}[T^A \bar{N}, H_{\bar{N}}]$. Clearly, (A, H) is a functor. Thus, we have proved

Proposition. *The above construction establishes a bijection between the bundle functors on $\mathcal{M}f_m \times \mathcal{M}f$ that preserve products in the second factor and have order r in the first factor and the pairs (A, H) of a Weil algebra and a group homomorphism $H\colon G^r_m \to \mathrm{Aut}\,A$.*

In particular, if $M = \mathbb{R}^m$, then $P^r\mathbb{R}^m = \mathbb{R}^m \times G^r_m$ and the associated bundle $(A,H)(\mathbb{R}^m, N)$ is identified with $\mathbb{R}^m \times T^A N$. Given $f\colon N \to \bar{N}$, we have

$$(A,H)(\mathrm{id}_{\mathbb{R}^m}, f) = \mathrm{id}_{\mathbb{R}^m} \times T^A f\colon \mathbb{R}^m \times T^A N \to \mathbb{R}^m \times T^A \bar{N}\,. \tag{5.10}$$

If $\bar{F} = (\bar{A}, \bar{H})$ is another bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ of order r in the first factor, then the natural transformations $\tau\colon F \to \bar{F}$ are in bijection with the equivariant algebra homomorphisms $\mu\colon A \to \bar{A}$, i.e. $\mu(H(X)(a)) = \bar{H}(X)(\mu(a))$ for all $a \in A$ and $X \in G^r_m$. We have

$$\tau_{M,N} = (\mathrm{id}_{P^r M}, \mu_N)\colon P^r M[T^A N, H_N] \to P^r M[T^{\bar{A}} N, \bar{H}_N]\,. \tag{5.11}$$

In the case $M = \mathbb{R}^m$, $\tau_{\mathbb{R}^m,N} = \mathrm{id}_{\mathbb{R}^m} \times \mu_N\colon \mathbb{R}^m \times T^A N \to \mathbb{R}^m \times T^{\bar{A}} N$.

Example. We describe the functor $\widetilde{J}^r$ in this way. The space $\widetilde{T}^r_m N := \widetilde{J}^r_0(\mathbb{R}^m, N)$ is called the bundle of nonholonomic (m,r)-velocities on N. Using the translations on $\mathbb{R}^m$, one identifies $\widetilde{T}^r_m N$ with the iteration $T^1_m(\dots(T^1_m N)\dots)$. Since the Weil algebra of T^1_m is $\mathbb{D}^1_m$, Section 2.9 implies that the Weil algebra $\widetilde{\mathbb{D}}^r_m$ of $\widetilde{J}^r$ is $\mathbb{D}^1_m \underbrace{\otimes\cdots\otimes}_{r\text{-times}} \mathbb{D}^1_m$. The action of $G^r_m = \mathrm{inv}\, J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ on $\widetilde{\mathbb{D}}^r_m$ is given by the composition of nonholonomic jets.

In general, consider a bundle functor F on $\mathcal{FM}$. The definition of the order of F can be based on the concept of (q,s,r)-jet, see Section 4.7.

Definition. We say that F is of order (q,s,r), $s \ge q \le r$, if for every $\mathcal{FM}$-morphism $f\colon Y \to \bar{Y}$, the restriction $Ff \mid F_y Y$ depends on $j^{q,s,r}_y f$ only, $y \in Y$. The integer r is called the base order of F.

5.5 One equivariant algebra homomorphism

Let F be a fiber product preserving bundle (in short: f.p.p.b.) functor on $\mathcal{FM}_m$, i.e. $F(Y_1 \times_M Y_2) = FY_1 \times_M FY_2$ for every two fibered manifolds Y_1 and Y_2 over the same base M, $\dim M = m$. We need the following result from analysis.

Lemma. *Every fiber product preserving bundle functor F on $\mathcal{FM}_m$ has finite order.*

Proof. Since the associated functor G^F preserves products, it has finite order r_1 by Theorem 2.4. We construct a natural bundle $B^F : \mathcal{M}f_m \to \mathcal{FM}$ by setting

$$B^F(M) = F(M \times \mathbb{R}), \quad B^F(f) = F(f \times \mathrm{id}_{\mathbb{R}}).$$

By [KMS], Chapter V, B^F has finite order r_2. We are going to deduce that the order of F is $\max(r_1, r_2)$. By locality of F, we may consider an $\mathcal{FM}_m$-morphism $f : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^n$ over $\underline{f} : \mathbb{R}^m \to \mathbb{R}^m$. Write $f_i : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}$ for the i-th fiber component of f, $i = 1, 2, \dots, n$. Then f is the fiber product of $f_1, \dots, f_n$ over $\underline{f}$. Since F preserves fiber products, it suffices to prove that F is of the order $\max(r_1, r_2)$ on $f_1 : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}$. We have $f_1 = (\underline{f}, \varphi)$, where $\varphi : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}$ is a map. Write $d : \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^k$ for the $\mathcal{FM}_m$-morphism $(x, y) \to (x, x, y)$. Then we have

$$f_1 = (\underline{f} \times \mathrm{id}_{\mathbb{R}}) \circ (\mathrm{id}_{\mathbb{R}^m} \times \varphi) \circ d.$$

Applying F at $0 \in \mathbb{R}^m$, we obtain

$$F_0 f_1 = B^F_0(\underline{f}) \circ G^F(\varphi) \circ F_0 d.$$

But $F_0 d$ is independent of f_1, so that $F_0 f_1$ depends on $j^{r_2}_0 \underline{f}$ and the r_1-jets of φ only. □

In several problems it is reasonable to consider the base order r of F. This is based on the following lemma.

Lemma (a). *If F is a f.p.p.b. functor on $\mathcal{FM}_m$ of the base order r, then the induced functor $F \circ i$ on $\mathcal{M}f_m \times \mathcal{M}f$ preserves products in the second factor and is of the order r in the first factor.*

Proof. Clearly, $F \circ i$ preserves products in the second factor. Further, consider the product maps $g_1 \times f$, $g_2 \times f : M \times N \to \bar{M} \times \bar{N}$, $f : N \to \bar{N}$. We have $j^r_x g_1 = j^r_x g_2$ by asssumption. Hence $j^r_{x,y}(g_1, f) = (j^r_x g_1, j^r_y f) = (j^r_x g_2, j^r_y f) = j^r_{x,y}(g_2, f)$, i.e. (g_1, f) and (g_2, f) have contact of order r at $(x, y) \in M \times N$. □

Thus, $F \circ i = (A, H)$, so that $F(M \times N)$ is the associated bundle

$$F(M \times N) = P^r M[T^A N, H_N]. \tag{5.12}$$

The left action ϱ of G^r_m on $T^A N$ is

$$\varrho(j^r \circ \varphi)(j^A \psi) = j^A(\varphi \circ \varphi^{-1}), \quad \varphi : \mathbb{R}^n \to \mathbb{R}^m, \quad \psi : \mathbb{R}^m \to N. \tag{5.13}$$

Further, F determines a natural transformation $\widetilde{t}_Y\colon J^r_h Y \to FY$. Every element $X \in J^r_h Y$ is of the form $j^r_x s$. We interpret the local section s of Y as a local $\mathcal{FM}_m$-morphism $\widetilde{s}$ of the trivial fibered manifold $\mathrm{id}_M\colon M \to M$ into $Y \to M$ and we set

$$\widetilde{t}_Y(X) = (F\widetilde{s})(x) \in FY\,.$$

In the product case $Y = \mathbb{R}^m \times N$, we have $J^r_h(\mathbb{R}^m \times N) = \mathbb{R}^m \times T^r_m N$ and $F(\mathbb{R}^m \times N) = \mathbb{R}^m \times T^A N$. Write

$$t_N\colon T^r_m N \to T^A N$$

for the restricted and corestricted map over $0 \in \mathbb{R}^m$. This is a natural transformation, so that it corresponds to an algebra homomorphism $t\colon \mathbb{D}^r_m \to A$. By naturality of $\widetilde{t}$, t is a G^r_m-equivariant algebra homomorphism

$$t(X \circ g) = H\big(g^{-1}(t(X))\big)\,, \quad X \in \mathbb{D}^r_m,\ g \in G^r_m\,.$$

Consider a fibered manifold $p\colon Y \to M$ and the injection of Y into $M \times Y$

$$I\colon Y \hookrightarrow M \times Y\,, \quad y \mapsto \big(p(y), y\big)\,.$$

Applying F, we obtain an injection

$$FI\colon FY \hookrightarrow F(M \times Y) = P^r M[T^A Y, H_Y]\,. \tag{5.14}$$

Clearly, $P^r M \subset T^r_m M$. Using (5.12), we obtain easily
Lemma (b). *For every $u \in P^r M$ and $Z \in T^A Y$, we have $\{u, Z\} \in FY$ if and only if*

$$t_M(u) = T^A_p(Z) \in T^A M\,. \qquad \square \tag{5.15}$$

Hence every f.p.p.b. functor F on $\mathcal{FM}_m$ determines a triple (A, H, t).

5.6 The main result

Conversely, consider a triple (A, H, t), where $H\colon G^r_m \to \mathrm{Aut}\,A$ is a group homomorphism and $t\colon \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism, $\mathrm{Aut}\,\mathbb{D}^r_m = G^r_m$. Such a triple defines a f.p.p.b. functor $F = (A, H, t)$ on $\mathcal{FM}_m$ by

$$FY = \big\{\{u, Z\}, u \in P^r M, Z \in T^A Y, t_M(u) = T^A p(Z)\big\}\,. \tag{5.16}$$

For an $\mathcal{FM}_m$-morphism $f\colon Y \to \bar{Y}$ over $\underline{f}\colon M \to \bar{M}$, $(A, H)(\underline{f}, f) = P^r \underline{f}[T^A f]$ maps FY into $F\bar{Y}$ and Ff is its restriction and corestriction. This implies

Theorem. *The f.p.p.b. functors of base order r on $\mathcal{FM}_m$ are in bijection with the triples (A, H, t), where A is a Weil algebra, $H\colon G^r_m \to \operatorname{Aut} A$ is a group homomorphism and $t\colon \mathbb{D}^r_m \to A$ is an equivariant algebra homomorphism. The natural transformations are in bijection with the equivariant algebra homomorphisms $\mu\colon A \to \bar{A}$ satisfying $\bar{t} = \mu \circ t$.*

The second assertion follows from the fact that $\bar{t} = \mu \circ t$ implies that the natural transformation $(A, H) \to (\bar{A}, \bar{H})$ maps $(A, H, t)(Y)$ into $(\bar{A}, \bar{H}, \bar{t})(Y)$.
If we use the inclusions $J^r_h Y \subset P^r M[T^r_m Y]$ and $FY \subset P^r M[T^A Y]$, then the map $\widetilde{t}_Y : J^r_h Y \to FY$ is of the form

$$\widetilde{t}_Y(\{u, X\}) = \{u, t_Y(X)\}\,. \tag{5.17}$$

The basic examples. The simpliest examples of f.p.p.b. functors on $\mathcal{FM}_m$ are J^r_h, J^r_v and V^A. So every iteration of these functors is also a f.p.p.b. functor on $\mathcal{FM}_m$.
By Section 1.8, $\operatorname{Aut} \mathbb{D}^r_m = G^r_m$. Write C for the corresponding action of G^r_m on $\mathbb{D}^r_m$. One finds easily $J^r_h = (\mathbb{D}^r_m, C, \mathrm{id}_{\mathbb{D}^r_m})$ and $J^r_v = (\mathbb{D}^r_m, C, \mathcal{O})$, where $\mathcal{O}\colon \mathbb{D}^r_m \to \mathbb{D}^r_m$ is the zero homomorphism. Functor V^A has base order 0. In this case, $G^r_m = \{e\}$ is the one-element group, $\mathbb{D}^0_m = \mathbb{R}$ and H maps e into $\mathrm{id}_\mathbb{R}$. We may always consider for t the zero homomorphism $\mathbb{D}^r_m \to \mathbb{D}^r_m$. This defines the so-called vertical version of (A, H).

Remark. The description of the composition of two f.p.p.b. functors on $\mathcal{FM}_m$ requires the theory of F-prolongations of principal fiber bundles. This will be explained in Section 6.6 below.

Further, the basic subject of this chapter are the f.p.p.b. functors on the category $\mathcal{FM}_m$. The main reason is that these functors have a great number of applications. It is worth mentioning that the problem of characterizing the f.p.p.b. functors on the category $\mathcal{FM}$ of all morphisms of fibered manifolds appeared not sooner as in 2006. Its solution can be found in [KoMi06].

5.7 Example

As an example, we construct a simple exchange map related with applications. Consider $F = (A, H, t)$ of base order r and the vertical Weil functor V^B. The base order of both $V^B F$ and $F V^B$ is r. In the first or the second case, the action of G^r_m on $A \otimes B$ or $B \otimes A$ is $H \otimes \mathrm{id}_B$ or $\mathrm{id}_B \otimes H$ and the algebra homomorphism $\mathbb{D}^r_m \to A \otimes B$ or $\mathbb{D}^r_m \to B \otimes A$ is $t \otimes \mathrm{id}_B$ or $\mathrm{id}_B \otimes t$, respectively. Hence the exchange algebra homomorphism $ex\colon A \otimes B \to B \otimes A$ is equivariant and satisfies $ex \circ (t \otimes \mathrm{id}_B) = \mathrm{id}_B \otimes t$. Thus, ex determines a canonical natural equivalence

$$\varkappa_Y^{B,F} \colon V^B(FY) \to F(V^B Y)\,. \tag{5.18}$$

The special case $F = J^r_h$ and $V^B = V$ is heavily used e.g. in the variational calculus on fibered manifolds.

Remark. On the other hand we have a quite different situation in the case of two horizontal jet prolongation functors J^r_h and J^s_h. In [DoKo01] it is deduced that the only natural transformation $J^r_h \circ J^s_h \to J^r_h \circ J^s_h$ is the identity.

5.8 The flow natural map

In the case of a f.p.p.b. functor $F = (A, H, t)$ on $\mathcal{FM}_m$, we have the following analogy of the flow natural map from Section 2.10. Consider first a vector field ξ on M. Its flow prolongation $P^r\xi$ is a right invariant vector field on the r-th order frame bundle $P^r M$ of M, whose value at every $u \in P^r_x M$ depends on $j^r_x \xi$ only. This defines a map

$$\nu^r_M \colon P^r M \times_M J^r TM \to TP^r M\,. \tag{5.19}$$

For a fibered manifold $p\colon Y \to M$, we will consider TY as a fibered manifold $TY \to M$. Then $Tp\colon TY \to TM$ is a base preserving morphism, that induces $FTp\colon FTY \to FTM$. Taking into account the natural transformation $\widetilde{t}_{TM} \colon J^r TM \to FTM$, we construct the fiber product

$$J^r TM \times_{FTM} FTY\,. \tag{5.20}$$

By (5.16), we have $FTY \subset P^r M[T^A TY]$. Consider an element $(X, \{u, Z\})$ from (5.20), $X \in J^r_x TM$, $u \in P^r_x M$, $Z \in T^A TY$. Write $\nu^r_M(u, X) = \big(\frac{\partial}{\partial t}\big)_0 \gamma(t)$, $\gamma\colon \mathbb{R} \to P^r M$. By (5.18), $\varkappa^A_Y(Z) \in TT^A Y$ can be expressed as $\big(\frac{\partial}{\partial t}\big)_0 \xi(t)$, where $\xi\colon \mathbb{R} \to T^A Y$ satisfies $t_M(\gamma(t)) = T^A p(\xi(t))$ for all t. So $\{\gamma(t), \xi(t)\}$ is a curve on FY and we define

$$\psi^F_Y\big(X, \{u, Z\}\big) = \frac{\partial}{\partial t}\Big|_0 \big\{\gamma(t), \xi(t)\big\}\,. \tag{5.21}$$

By right-invariancy, this is independent of the choice of u. Hence we obtain a map

$$\psi^F_Y \colon J^r TM \times_{FTM} FTY \to TFY\,. \tag{5.22}$$

By (5.21), FTY is a vector bundle over $FTM \times_M FY$ and one finds easily that ψ^F_Y is linear in both J^rTM and FTY.
A projectable vector field η on Y over ξ on M can be interpreted as a base-preserving morphism $\eta\colon Y \to TY$. Then we construct the functorial prolongation $F\eta\colon FY \to FTY$ as well as the r-th jet prolongation $j^r\xi\colon M \to J^rTM$. The values of $j^r\xi \times_{\mathrm{id}_M} F\eta$ are in (5.20).

Proposition. *The flow prolongation $\mathcal{F}\eta$ of η satisfies*

$$\mathcal{F}\eta = \psi^F_Y \circ (j^r\xi \times_{\mathrm{id}_M} F\eta)\,. \tag{5.23}$$

Proof. In general, for a principal bundle $P(M,G)$ and an associated bundle $P[S]$, a right-invariant vector field ξ on P and a left-invariant vector field σ on S induce a vector field $\{\xi,\sigma\}$ on $P[S]$, see Section 3.8. In our case, the flow construction implies that the vector field

$$\big\{P^r\xi, \mathcal{T}^A\eta\big\}\,,\quad P^r\xi\colon P^rM \to TP^rM\,,\quad \mathcal{T}^A\eta\colon T^AY \to TT^AY \tag{5.24}$$

on $P^rM[T^AY]$ is restrictible to the submanifold FY and this restriction coincides with $\mathcal{F}\eta$. According to (5.16), the functorial prolongation $F\eta\colon FY \to FTY$ of $\eta\colon Y \to TY$ is of the form $\mathrm{id}_{P^rM}[T^A\eta]$. Since (5.19) identifies $j^r\xi$ with $\mathcal{P}^r\xi$, we have

$$j^r\xi \times_{\mathrm{id}_M} F\eta\colon FY \to J^rTM \times_{FTM} FTY\,.$$

Then (5.23) follows directly from (5.24). □

It is useful to introduce also a modified map

$$\begin{aligned} &\widetilde{\psi}^F_Y\colon J^rTM \times_{FTM} FTY \to J^rTM \times_{TM} TFY\,,\\ &\widetilde{\psi}^F_Y(X,\{u,Z\}) = \big(X, \psi^F_Y(X,\{u,Z\})\big)\,. \end{aligned} \tag{5.25}$$

By construction, $\widetilde{\psi}^F_Y$ is a diffeomorphism.
In the case $F = J^r$, we have $FTM = J^rTM$, so that

$$\psi^F_Y\colon J^rTY \to TJ^rY\,.$$

This map was constructed in another way by L. Mangiarotti and M. Modugno, [MM83].
We present a local expression of ψ^F_Y. We take $Y = M \times Q$, where we may assume Q is a vector space. The group homomorphism $H\colon G^r_m \to \mathrm{Aut}\,A$

induces a Lie algebra homomorphism $\mathfrak{h}\colon \mathfrak{g}_m^r \to \operatorname{Der} A$. We have $J^r T\mathbb{R}^m = T\mathbb{R}^m \times \mathfrak{g}_m^r$. Since $\widetilde{t}_{T\mathbb{R}^m}$ maps $\mathfrak{g}_m^r$ into $(N_A)^m$, it is

$$J^r T\mathbb{R}^m \times_{FT\mathbb{R}^m} FT(\mathbb{R}^m \times Q) = T\mathbb{R}^m \times \mathfrak{g}_m^r \times Q \otimes A \times Q \otimes A\,.$$

On the other hand, $TF(\mathbb{R}^m \times Q) = T\mathbb{R}^m \times Q \otimes A \times Q \otimes A$. For $z \in T\mathbb{R}^m$, $u \in \mathfrak{g}_m^r$, $v \otimes a \in Q \otimes A$ and $w \in Q \otimes A$, one finds

$$\psi^F_{\mathbb{R}^m \times V}(z, u, v \otimes a, w) = \big(z, v \otimes a, w + v \otimes \mathfrak{h}(u)(a)\big)\,, \quad \text{see [Ko03]}\,. \tag{5.26}$$

Even in the case $F = J^r$, this formula is convenient for evaluating $\mathcal{J}^r \eta$.

5.9 The case of vertical Weil bundles

We describe the special case of the vertical B-prolongation $V^B(V^A Y \to M)$ of the vertical A-prolongation of a fibered manifold $p\colon Y \to M$. From the manifold case, we know the exchange map

$$\varkappa_Y^{B,A}\colon T^B(T^A Y) \to T^A(T^B Y)\,.$$

Lemma. *$\varkappa_Y^{B,A}$ maps $V^B(T^A Y \to T^A M)$ into $T^A(V^B Y)$.*

Proof. By locality, it suffices to consider a product bundle $Y = (M \times Q) \to M$. We have

$$T^A Y = T^A M \times T^A Q\,, \quad V^B(T^A Y \to T^A M) = T^A M \times T^B T^A Q\,,$$
$$V^B Y = M \times T^B Q\,, \quad T^A(V^B Y) = T^A M \times T^A T^B Q\,.$$

In this situation, $\varkappa_Y^{B,A}$ is reduced to the exchange diffeomorphism $\varkappa_Q^{B,A}\colon T^B T^A Q \to T^A T^B Q$. □

The restricted and corestricted map, that will be denoted by

$$\varkappa_{Y,V}^{B,A}\colon V^B(V^A Y \to M) \to V^A(V^B Y \to M)\,, \tag{5.27}$$

represents the exchange diffeomorphism applied fiberwise. For $B = \mathbb{D}$, we write

$$\varkappa_{Y,V}^{A}\colon V(V^A Y \to M) \to V^A(VY \to M)\,. \tag{5.28}$$

Let η be a vertical vector field on Y and $\mathcal{V}^A \eta$ be its flow prolongation. Analogously to (2.10), we obtain

$$\mathcal{V}^A \eta = \varkappa_{Y,V}^{A} \circ V^A \eta\,. \tag{5.29}$$

Chapter 6

On the geometry of (A, H, t)-prolongations

6.1 Prolongation of vector bundles

Consider a vector bundle $p\colon E \to M$. The vector addition in E and the multiplication of vectors by real numbers are two maps

$$a\colon E \times_M E \to E\,, \quad m\colon \mathbb{R} \times E \to E\,. \tag{6.1}$$

For a Weil functor T^A, we clarified in Section 3.7 that $T^A p\colon T^A Y \to T^A M$ is also a vector bundle. Further, if $\overline{p}\colon \overline{E} \to \overline{M}$ is another vector bundle and $f\colon E \to \overline{E}$ is a linear morphism with base map $\underline{f}\colon M \to \overline{M}$, then $T^A f\colon T^A E$ is a linear morphism over $T^A \underline{f}\colon T^A M \to T^A \overline{M}$.
In the case of $F = (A, H, t)$, we can apply F directly to the first formula in (6.1). This yields

$$Fa\colon FE \times_M FE \to FE \tag{6.2}$$

with the fiber product over M. Further we have to take into account that F can be applied to fibered manifolds only. Hence we rewrite the second formula from (6.1) as

$$m\colon (M \times \mathbb{R}) \times_M E \to E\,.$$

Then (5.12) implies

$$Fm\colon P^r M[A] \times_M FE \to FE \tag{6.3}$$

with the linear action of G^r_m on $A = \mathbb{R} \times N$ determined by H. But $\mathbb{R}$ is invariant with respect to $H(g)$ for all $g \in G^r_m$, so that $M \times \mathbb{R} \subset P^r M[A]$.

Restricting (6.3) to this subset, we obtain the multiplication by real scalars on FE. A direct discussion of all these constructions implies

Proposition. *$FE \to M$ is also a vector bundle.*

Analogously to the case of T^A, we deduce

Corollary. *For every linear morphism $f\colon E \to \overline{E}$ over a local diffeomorphism $\underline{f}\colon M \to \overline{M}$, $Ff\colon FE \to F\overline{E}$ is a linear morphism over $\underline{f}$.*

6.2 Group bundles

It is well known that the r-th jet prolongation J^rP of a principal bundle $P(M,G)$ does not have a natural principal bundle structure over M, [KMS]. Consider the general case $F = (A, H, t)$. If we want to apply F, we have to interpret the right action $\varrho\colon P \times G \to P$ as a fibered manifold morphism

$$\varrho\colon P \times_M (M \times G) \to P\,. \tag{6.4}$$

By (5.9), $F(M \times G) = P^rM[T^AG]$ and the F-prolongation of (6.4) yields

$$F\varrho\colon FP \times_M P^rM[T^AG] \to FP\,, \tag{6.5}$$

where, in general, $P^rM[T^AG]$ is globally not a product of M with a Lie group.
In this situation, it is useful to introduce the general concept of a group bundle. A fibered manifold $p\colon K \to M$ is called a group bundle, if each fiber K_x, $x \in M$ is a Lie group and K is locally trivial in the following sense: there is a Lie group C and a neighbourhood U of every $x \in M$ such that $p^{-1}(U) \approx U \times C$.
Hence the group composition forms a base preserving morphism $\nu\colon K \times_M K \to K$. The product $M \times C$ is called the product group bundle. Clearly, for every group bundle $K \to M$, $FK \to M$ is also a group bundle. Indeed, by (6.5), $F(M \times C) = P^rM[T^AC]$ and

$$F\nu = \mathrm{id}_{P^rM}[T^A\nu]\,. \tag{6.6}$$

Let $\mathrm{Aut}\,C$ be the group of all automorphisms of the group C. Consider a principal bundle $Q(M,G)$ and a group homomorphism $\varphi\colon G \to \mathrm{Aut}\,C$. This defines a left action $(g,c) \mapsto \varphi(g^{-1})(c)$ of G on C, so that we can construct the associated bundle $Q[C]$. This is a group bundle, if we define

$$\{u,h_1\}\{u,h_2\} = \{u,h_1h_2\}\,, \quad u \in Q,\ h_1,h_2 \in H\,. \tag{6.7}$$

6.3 Weak principal bundles

The following concept is very important for the theory of (A, H, t)-prolongations.

Definition. A fibered manifold $Q \to M$ is called a weak principal bundle with structure group bundle $K \to M$, if we are given a base preserving morphism $\varrho\colon Q \times_M K \to Q$ such that each group K_x acts simply transitively on the right on Q_x.

Clearly, every principal bundle $Q(M, G)$ is a weak principal bundle, whose structure group bundle is the product $M \times G$.
An important fact is that the (A, H, t)-prolongations of weak principal bundles, $F = (A, H, t)$, are weak principal bundles again.

Proposition. *If $\varrho\colon Q \times_M K \to Q$ is a weak principal bundle with structure group bundle $K \to M$, then $F\varrho\colon FQ \times_M FK \to FQ$ is a weak principal bundle with structure group bundle $FK \to M$.*

Proof. Locally, we may assume $Q = \mathbb{R}^m \times S$, $K = \mathbb{R}^m \times G$ and consider a right action σ of G on S such that

$$\varrho\big((x, s), (x, g)\big) = \big(x, \sigma(s, g)\big)\,, \quad x \in \mathbb{R}^m,\, s \in S,\, g \in G\,. \tag{6.8}$$

By Chapter 5, $F\varrho\colon FQ \times_M FK \to FQ$ is of the form

$$F\varrho\big((x, s_1), (x, g_1)\big) = \big(x, T^A\sigma(s_1, g_1)\big)\,, \quad s_1 \in T^A S, g_1 \in T^A G\,. \tag{6.9}$$

□

6.4 Principal prolongations of principal bundles

Consider a principal bundle $P(M, G)$ and write $\varphi\colon G \times G \to G$ for the group composition and $\varrho\colon P \times G \to P$ for the right action of G on P. By Chapter 3, G^r_m acts on $T^r_m G$ by isomorphisms. Hence we can construct the semidirect group product $W^r_m G = G^r_m \rtimes T^r_m G$,

$$(g_1, X_1)(g_2, X_2) = \big(g_1 \circ g_2, T^r_m\varphi(X_1 \circ g_2, X_2)\big)\,. \tag{6.10}$$

Then $W^r P := (P^r M \times_M J^r P) \to M$ is a principal bundle with structure group $W^r_m G$ and its right action on $W^r P$ is

$$(u, V)(g, Z) = \big(u \circ g, T^r_m\varrho(Z, V)\big)\,. \tag{6.11}$$

$u \in P^r M$, $g \in G^r_m$, $V \in J^r P$, $Z \in T^r_m G$. We say that $W^r P(M, W^r_m G)$ is the r-th principal prolongation of the principal bundle $P(M, G)$.
An action $\ell \colon G \times S \to S$ induces an action $W^r_m \ell \colon W^r_m G \times T^r_m S \to T^r_m S$

$$W^r_m \ell\big((g, X), Z\big) = \big(T^r_m \ell(X, Z)\big) \circ g^{-1}\,, \tag{6.12}$$

$g \in G^r_m$, $X \in T^r_m G$, $Z \in T^r_m S$. Then $J^r E$ has a canonical structure of the associated bundle

$$J^r E = W^r P[T^r_m S, W^r_m \ell]\,. \tag{6.13}$$

Consider the r-th principal prolongation $W^r(P^s M)$ of the s-th order frame bundle $P^s M$ of a manifold M. Every local diffeomorphism $\varphi \colon \mathbb{R}^m \to M$ induces a principal fiber bundle morphism $P^s\varphi \colon P^s\mathbb{R}^m \to P^s M$ and we can construct $j^r_{(0,e_s)}(P^s\varphi) \in W^r(P^s M)$, where e_s denotes the unit of G^s_m. One sees directly that this element depends on the $(r+s)$-jet $j^{r+s}_0\varphi$ only. Hence the map $j^{r+s}_0\varphi \mapsto j^r_{(0,e_s)}(P^s\varphi)$ defines an injection

$$i_M \colon P^{r+s}M \to W^r(P^s M)\,. \tag{6.14}$$

Since the group multiplication in both G^{r+s}_m and $W^r_m G^s_m$ is defined by the composition of jets, the restriction $i_0 \colon G^{r+s}_m \to W^r_m G^s_m$ of $i_{\mathbb{R}^m}$ to the fibers over $0 \in \mathbb{R}^m$ is a group homomorphism. Thus, the $(r+s)$-order frames on a manifold M form a natural reduction $i_M \colon P^{r+s}M \to W^r(P^s M)$ of the r-th principal prolongation of the s-th order frame bundle of M to the subgroup $i_0(G^{r+s}_m) \subset W^r_m G^s_m$.

6.5 (A, H, t)-prolongations of principal and associated bundles

In the general case of $F = (A, H, t)$, consider first a weak principal bundle $\varrho_Q \colon Q \times_M K \to Q$ with structure group bundle $K \to M$. By Section 6.3,

$$F\varrho_Q \colon FQ \times_M FK \to FQ$$

is a weak principal bundle with structure group bundle $FK \to M$.
Even the construction of $W^r P$ from Section 6.4 can be extended to the case of $F = (A, H, t)$. By Section 6.4, the action H_G of G^r_m on $T^A G$ is by group homomorphisms. Hence we can construct the semidirect group product $W^A_H G = G^r_m \rtimes T^A G$ with the composition

$$(g_1, X_1)(g_2, X_2) = \big(g_1 \circ g_2, T^A\varphi(H_G(g_2^{-1})(X_1), X_2)\big)\,. \tag{6.15}$$

For a left action $\ell\colon G \times S \to S$, we define

$$W_H^A\ell\colon W_H^A G \times T^A S \to T^A S\,, \quad W_H^A\ell\big((g, X), Z\big) = H_S(g)\big(T^A\ell(X, Z)\big)\,, \tag{6.16}$$

$g \in G_m^r$, $X \in T^A G$, $Z \in T^A S$. This is an action, too. Clearly,

$$F(M \times G) = P^r M[T^A G, H_G]$$

is a group bundle of type $T^A G$. We have

$$F\varrho_P\colon FP \times_M P^r M[T^A G, H_G] \to FP\,.$$

We introduce $W^F P = P^r M \times_M FP$ and we define a right action of $W_H^A G$ on $W^F P$ by

$$(u, Z)(g, X) = \big(u \circ g, F\varrho_P(Z, \{u \circ g, X\})\big)\,, \tag{6.17}$$

$u \in P_x^r M$, $Z \in F_x P$, $g \in G_m^r$, $X \in T^A G$, so that $\{u \circ g, X\} \in P^r M[T^A G, H_G]$. Then one verifies directly

Proposition. *$W^F P(M, W_H^A G)$ is a principal bundle. For an associated bundle $E = P[S, \ell]$, FE is an associated bundle $W^F P[T^A S, W_H^A \ell]$.*

We say that $W^F P$ is the principal F-prolongation of principal bundle P. In the case $F = J^r$, we have $W^{J^r} P = W^r P$.

6.6 The composition of two functors

First we deduce two results about Weil algebras. We know $T^A B = A \otimes B$. A direct description of the algebra multiplication is as follows. If $m\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the multiplication of reals, then the algebra multiplication in B is

$$T^B m\colon T^B\mathbb{R} \times T^B\mathbb{R} \to T^B\mathbb{R} \tag{6.18}$$

and the algebra multiplication in $T^A B$ is the T^A-prolongation of (6.18).

Lemma. *For every algebra homomorphism $h\colon B \to B$, $T^A h\colon T^A B \to T^A B$ is also an algebra homomorphism. For every algebra homomorphism $h\colon A \to \overline{A}$, the natural transformation $h_B\colon T^A B \to T^{\overline{A}} B$ is an algebra homomorphism.*

Proof. The multiplication in B or $\overline{B}$ is of the form (6.18) and h satisfies $T^B m \circ (h \times h) = h \circ T^B m$. Applying T^A to this equation, we prove the first assertion of the lemma. To deduce the second one, we apply h_B to the algebra multiplication in B. □

Consider another functor $E=(B,K,u)$. Our main aim is to express the iterated functor $F\circ E$ in the form (C,L,v). Clearly, $C=T^AB=A\otimes B$. The base order of $F\circ E$ is $r+s$, so that L is to be a group homomorphism $G^{r+s}_m\to \mathrm{Aut}(A\otimes B)$ and v is to be an algebra homomorphism $\mathbb{D}^{r+s}_m\to A\otimes B$. In order to determine L, we use the product expressions

$$E(M,N)=P^sM[T^BN,K_N] \tag{6.19}$$

and

$$(F\circ E)(M,N)=P^{r+s}M[T^{A\otimes B}N,L_N]\,.$$

By Proposition 6.5, we have

$$F\big(E(M,N)\big)=W^FP^sM\big[T^AT^BN,W^A_HK_N\big]\,.$$

According to the geometric construction of (6.17), we have to add the principal bundle morphism $W^rP\to W^FP$ determined by t

$$W^rP^sM\to W^FP^sM$$

and the inclusion $P^{r+s}M\hookrightarrow W^rP^sM$. At the group level, the action

$$W^A_HK_N\colon W^A_HG^s_m\times T^AT^BN\to T^AT^BN$$

is to be composed with the group homomorphism

$$t_{G^s_m}\colon W^r_mG^s_m\to W^A_HG^s_m$$

and the injection $i^{r,s}_m\colon G^{r+s}_m\to W^r_mG^s_m$.
The algebra homomorphism v can be reconstructed from the natural transformation $\widetilde{v}\colon J^{r+s}\to F\circ E$. Consider the classical injection

$$I^{r,s}_Y\colon J^{r+s}Y\to J^r(J^sY),\ \ I^{r+s}_Y(j^{r+s}_x\sigma)=j^r_x(j^s\sigma)\,,$$

where σ is a section of Y.
By the definition of $\widetilde{v}$, we have

$$\widetilde{v}_Y(j^{r+s}_x\sigma)=(F\circ E)(\sigma)(x)\,. \tag{6.20}$$

Applying J^r to the natural transformation $\widetilde{v}_Y\colon J^sY\to EY$, we obtain $J^r(\widetilde{v}_Y)\colon J^rJ^sY\to J^rEY$. Then the right-hand side of (6.20) can be rewritten as

$$\widetilde{t}_{EY}\big(J^r(\widetilde{u}_Y)\big)\big(I^{r,s}_Y(j^{r+s}_x\sigma)\big)\,.$$

Hence

$$\widetilde{v}_Y = (\widetilde{t}_{EY}) \circ J^r(\widetilde{u}_Y) \circ I^{r,s}_Y \,. \tag{6.21}$$

At the algebra level, we have a canonical injection

$$\iota^{r,s}_{m,\mathbb{D}} \colon \mathbb{D}^{r+s}_m \to T^r_m \mathbb{D}^s_m \,, \; j^{r+s}_0 f \mapsto j^r_0\big(j^s_y(f \circ \tau_{f(y)})\big) \,, \quad y \in \mathbb{R}^m \,.$$

Further, we construct the algebra homorphism $T^r_m u \colon T^r_m \mathbb{D}^s_m \to T^r_m B$ and the natural transformation $t_B \colon T^r_m B \to T^A B$. Then (6.20) implies

$$v = t_B \circ T^r_m u \circ \iota^{r,s}_{m,\mathbb{D}} \,. \tag{6.22}$$

Thus, we have proved

Proposition. *We have $F \circ E = (A \otimes B, L, v)$, where $L = W^A_H K \circ t_{G^s_m} \circ \iota^{r,s}_m$ and $v = t_B \circ T^r_m u \circ \iota^{r,s}_{m,\mathbb{D}}$.*

In particular, this proposition gives an exchange formula of $F \circ E$ with $E \circ F$. Even though this formula is much more sophisticated to the case of $T^B \circ T^A$ and $T^A \circ T^B$ from Section 2.9, it yields several remarkable geometric results on the existence or non-existence of the natural exchanges of $F \circ E$ with $E \circ F$, [DoKo01].

Remark. Consider two bundle functors F and E on $\mathcal{FM}_m$. According to Doupovec, [Do05], E is said to be subordinated to F, if there exists a surjective natural transformation $F \to E$. The theory of these functors is somewhat more sofisticated to the Weilian case of Section 2.12. The basic properties of such functors are deduced in [Do05].

6.7 Contact elements on fibered manifolds

We recall that a contact A-element on a manifold M is the set $(\operatorname{Aut} A)(X)$, where $X \in \operatorname{reg} T^A M$, see Section 2.15.
Consider a fibered manifold $p \colon Y \to M$. For an algebra homomorphism $\mu \colon A \to B$, we define

$$\operatorname{reg} T^\mu Y = \operatorname{reg} T^A M \times_{T^B M} \operatorname{reg} T^B Y \,. \tag{6.23}$$

A morphism $\nu \colon \mu \to \mu$ is a pair of algebra homomorphisms

$$\nu_1 \colon A \to A \quad \text{and} \quad \nu_2 \colon B \to B \tag{6.24}$$

such that the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\mu} & B \\ {\scriptstyle \nu_1}\downarrow & & \downarrow{\scriptstyle \nu_2} \\ A & \xrightarrow{\mu} & B \end{array} \tag{6.25}$$

The group of all isomorphisms $\mu \to \mu$ will be denoted by $\operatorname{Aut} \mu$.

Definition. The bundle of contact elements of type μ on Y is the factor space $K^\mu Y = \operatorname{reg} T^\mu Y / \operatorname{Aut} \mu$.

We shall write $\mu \colon T^\mu Y \to K^\mu Y$ for the factor projection. The following assertion can be deduced analogously to Section 2.15.

Proposition. *There is a unique manifold structure on $K^\mu Y$ such that the factor projection $\varkappa \colon \operatorname{reg} T^\mu Y \to K^\mu Y$ is a submersion.*

Corollary. $\operatorname{reg} T^\mu Y \to K^\mu Y$ *is a principal fiber bundle with structure group* $\operatorname{Aut} \mu$.

6.8 (A, H, t)-prolongation of general connections

First we recall the fundamental procedure of prolongating a general connection Γ on $p \colon Y \to M$ with respect to a bundle functor F on $\mathcal{FM}_m$, [KMS], p. 363. To clarify the basic ideas, we present a slight modification of that consideration.
Let F be a functor on $\mathcal{FM}_m$ and $\mathcal{F}$ denote its flow operator.

Lemma. *If the order of F is r and η is a projectable vector field on Y, then the value $(\mathcal{F}\eta)(u)$ at every $u \in (FY)_y$ depends on $j^r_y \eta$ only. The induced map*

$$\mathcal{F}\Gamma \colon FY \times_M J^r TY \to T(FY) \tag{6.26}$$

is smooth and linear with respect to $J^r TY = J^r(TY \to M)$.

Proof. Smoothness is direct consequence of our assumptions. Linearity follows immediately from the linearity of the flow operator $\mathcal{F}$. □

Consider an auxiliary r-th order linear connection Λ on TM, i.e a linear splitting

$$\Lambda \colon TM \to J^r TM \, . \tag{6.27}$$

The flow prolongation of the lifted vector field ΓX depends on the r-jet of vector field $X \colon M \to TM$. This defines a map

$$\mathcal{F}\Gamma \colon FY \times_M J^r TM \to TFY \, . \tag{6.28}$$

If we add $\Lambda\colon TM \to J^rTM$, we obtain a map

$$\mathcal{F}(\Gamma, \Lambda)\colon FY \times_M TM \to TFY\,. \tag{6.29}$$

This is a general connection $\mathcal{F}(\Gamma, \Lambda)$ on $FY \to M$ called the F-prolongation of Γ with respect to Λ.

Remark. The connection $\mathcal{F}(\Gamma, \Lambda)$ can be also constructed by using the flow natural map from Section 5.8.

Further we remark that some interesting applications concerning functorial prolongations of Lie algebroids and Lie groupoids can be found in [Ko05] and [Ko07].

6.9 An abstract characterization of the classical r-jets

Consider the homomorphic images of the germs of smooth maps. The classical construction of the r-th order jets transforms every pair M, N of manifolds into a fibered manifold $J^r(M, N)$ over $M \times N$ and every germ of a smooth map $f\colon M \to N$ at $x \in M$ into its r-jet $j^r(\mathrm{germ}_x f) = j^r_x f \in J^r(M, N)$. The composition $B_2 \circ B_1$ of germs induces the composition of r-jets (denoted by the same symbols)

$$j^r(B_2) \circ j^r(B_1) = j^r(B_2 \circ B_1)\,.$$

Hence the r-jets are finite-dimensional homomorphic images of the germs of smooth maps. First we point out some general properties of the pairs (J^r, j^r) for all $r \in \mathbb{N}$.
Denote by $G(M, N)$ the set of all germs of smooth maps from M into N. Consider a rule F transforming every pair of manifolds into a fibered manifold $F(M, N)$ over $M \times N$ and a system φ of maps $\varphi_{M,N}\colon G(M, N) \to F(M, N)$ commuting with the canonical projections $G(M, N) \to M \times N$ and $F(M, N) \to M \times N$ for all M, N. Then we can formulate the following requirements I–IV.

I. (Surjectivness) Every $\varphi_{M,N}\colon G(M, N) \to F(M, N)$ is surjective.

II. (Composability) If $\varphi(B_1) = \varphi(\overline{B}_1)$ and $\varphi(B_2) = \varphi(\overline{B}_2)$, then $\varphi(B_2 \circ B_1) = \varphi(\overline{B}_2 \circ \overline{B}_1)$.
By I and II, we have a well defined composition (denoted by the same symbol as the composition of germs or maps)

$$X_2 \circ X_1 = \varphi(B_2 \circ B_1)$$

for every $X_1 = \varphi(B_1) \in F_x(M,N)_y$ and $X_2 = \varphi(B_2) \in F_y(N,P)_z$. Let us write $\varphi_x f$ for $\varphi(\mathrm{germ}_x f)$. Every local diffeomorphism $f\colon M \to N$ and every smooth map $g\colon N \to \overline{N}$ induce a map $F(f,g)\colon F(M,N) \to F(\overline{M}, \overline{N})$ defined by

$$F(f,g)(X) = (\varphi_y g) \circ X \circ \varphi_{f(x)}(f^{-1}), \qquad X \in F_x(M,N)_y ,$$

where f^{-1} is constructed locally.

III. (Regularity) Each map $F(f,g)$ is smooth.

Consider a product $N_1 \xleftarrow{q_1} N_1 \times N_2 \xrightarrow{q_2} N_2$ of two manifolds. Then we have the induced maps $F(\mathrm{id}_M, q_1)\colon F(M, N_1 \times N_2) \to F(M, N_1)$ and $F(\mathrm{id}_M, q_2)\colon F(M, N_1 \times N_2) \to F(M, N_2)$. Both $F(M,N_1)$ and $F(M,N_2)$ are fibered manifolds over M.

IV. (Product property) $F(M, N_1 \times N_2)$ coincides with the fiber product $F(M,N_1) \times_M F(M,N_2)$ over M and $F(\mathrm{id}_M, q_1)$ and $F(\mathrm{id}_M, q_2)$ are the induced projections.

Theorem. *For every pair (F,φ) satisfying I–IV there exists an integer $r \geq 0$ such that $(F,\varphi) = (J^r, j^r)$.*

Proof. For every $k \in \mathbb{N}$, we define an induced functor F_k on $\mathcal{M}f$ by

$$F_k(M) = F_0(\mathbb{R}^k, M), \qquad 0 \in \mathbb{R}^k ,$$

and

$$F_k f = F_0(\mathrm{id}_{\mathbb{R}^k}, f)\colon F_k M \to F_k N$$

for every smooth map $f\colon M \to N$. By III, each map $F_k f$ is smooth and IV implies that each functor F_k preserves products.

Hence F_k is a Weil functor. Let $A_k = \mathbb{R} \times N_k$ be the corresponding algebra. By Section 1.15, A_k is a factor algebra $\mathcal{E}(k)/\mathcal{A}_k$, where $\mathcal{A}_k$ is an ideal of finite codimension, and the factor space $\mathcal{N}_k = \mathcal{A}_k/\mathfrak{m}(k)$ is the set of all germs $B \in G_0(\mathbb{R}^m, \mathbb{R})_0$ satisfying

$$\varphi(B) = \varphi(\,\widehat{0}\,),$$

where $\widehat{0}$ is the germ of the constant function 0. This yields the following "substitution property" of $\mathcal{N}_k$

$$B \in \mathcal{N}_k \quad \text{and} \quad h \in G_0(\mathbb{R}^k, \mathbb{R}^k)_0 \quad \text{implies } B \circ h \in \mathcal{N}_k . \tag{6.30}$$

Since $\mathcal{N}_k$ is an ideal of finite codimension, there exists an integer r_k such that

$$\mathfrak{m}(k)^{r_k+1} \subset \mathcal{N}_k\,. \tag{6.31}$$

Assume r_k is minimal. Then beside (6.31) we also have

$$\mathfrak{m}(k)^{r_k} \not\subset \mathcal{N}_k\,.$$

□

Now we deduce

Lemma. *The functor F_k is the velocities functor $T_k^{r_k}$.*

Proof. We have to prove

$$\mathfrak{m}(k)^{r_k+1} = \mathcal{N}_k\,.$$

We deduce the opposite inclusion to (6.31) by contradiction. We prove that if there exists an element $Q \notin \mathfrak{m}(k)^{r_k+1}$ satisfying $Q \in \mathcal{N}_k$, then the substitution property (6.30) implies $\mathfrak{m}(k)^{r_k} \subset \mathcal{N}_k$. Indeed, by (6.31) we may assume Q is a homogeneous polynomial of degree r_k. Let $b \in \mathbb{R}^k$ be a point such that $Q(b) \neq 0$. Consider the map $h\colon \mathbb{R}^k \to \mathbb{R}^k$, $h(x) = bx$, $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. Then $Q \circ h = Q(b)(x_1)^{r_k}$. By the substitution property, $Q(b)(x_1)^{r_k} \in \mathcal{N}_k$ and $Q(b) \neq 0$ implies $(x_1)^{r_k} \in \mathcal{N}_k$. Consider further the map $\overline{h}\colon \mathbb{R}^k \to \mathbb{R}^k$, $\overline{h}(x_1, \dots, x_k) = (c_1x_1 + \dots + c_kx_k, 0, \dots, 0)$ with arbitrary $c_1, \dots, c_k \in \mathbb{R}$. Then $(c_1x_1 + \dots + c_kx_k)^{r_k} \in \mathcal{N}_k$ for all $c_1, \dots c_k$. If we evaluate this expression and interpret $c_1, \dots, c_k$ as undetermined, we obtain that each monomial of degree r_k belongs to $\mathcal{N}_k$. Hence $\mathfrak{m}(k)^{r_k} \subset \mathcal{N}_k$. Next we use the following property of r-jets, [KMS]. If two germs $\mathrm{germ}_x f$, $\mathrm{germ}_x g \in G_x(M,N)_y$ satisfy $j_0^r(f \circ \delta) = j_0^r(g \circ \delta)$ for all $\delta\colon \mathbb{R} \to M$ with $\delta(0) = x$, then $j_x^r f = j_x^r g$. This is directly extended to the following assertion. If

$$j_0^r(f \circ \varepsilon) = j_0^r(g \circ \varepsilon) \quad \text{for all} \quad \varepsilon\colon \mathbb{R}^k \to M \quad \text{with} \quad \varepsilon(0) = x\,, \tag{6.32}$$

then $j_x^r f = j_x^r g$.
Next we prove

$$r_k = r_l \quad \text{for all} \quad k, l \in \mathbb{N}\,.$$

Indeed, by the above lemma we know that for every f, $g \in \mathbb{R}^k \to M$ the condition $\varphi_0 f = \varphi_0 g$ means $j_0^{r_k} f = j_0^{r_k} g$ with maximal r_k. For every $h \in G_0(\mathbb{R}^l, \mathbb{R}^k)_0$ we have $\varphi_0(f \circ h) = \varphi_0(g \circ h)$. By the definition of r_l this implies $j^{r_l}(f \circ h) = j^{r_l}(g \circ h)$. Hence $j_0^{r_l} f = j_0^{r_l} g$ by (6.32). Since r_k

is maximal, we have $r_k \geq r_l$. Replacing k and l, we obtain the converse relation $r_l \geq r_k$.
Denote by r the common value of all r_k. Then it suffices to deduce that each $F(M,N)$ is the associated fiber bundle $F(M,N) = P^r M[T^r_m N]$ to the r-th order frame bundle $P^r M(M, G^r_m)$ with respect to the jet action of its structure group G^r_m on $T^r_m N$, $m = \dim M$. For every $v \in P^r_x M$, $v = \varphi(V)$, we define $\widetilde{v}\colon F_x(M,N) \to T^r_m N$ by $\widetilde{v}(\varphi(B)) = \varphi(B \circ V)$. It holds $\varphi(B \circ V) = \varphi(B) \circ \varphi(V)$ and $\varphi(V) = j^r V$, $\varphi(B \circ V) = j^r(B \circ V)$. For $W = V \circ H$, $j^r H \in G^r_m$, we obtain $\widetilde{w}(\varphi(B)) = \varphi(B \circ V) \circ \varphi(H) = j^r(B \circ V) \circ j^r H$. Hence we have the standard situation of the smooth associated bundles. □

6.10 On the jets of foliation respecting maps

A foliated manifold $(M, \mathcal{F})$ is a manifold M endowed with a regular integrable distribution $\mathcal{F}$, [KMS]. The maximal integral manifold L_x passing through $x \in M$ is called a leaf of $\mathcal{F}$. Given two foliated manifolds $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$, a map $f\colon M_1 \to M_2$ is said to be a foliated morphism, if f maps the leafs of M_1 into the leafs of M_2. We denote by $\mathcal{F}ol$ the category of all foliated manifolds and all foliated morphisms.
Given two foliated manifolds $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$, a leafwise r-jet of M_1 into M_2 means an r-jet $j^r_x \gamma$ of a local map γ of L_x into $L_{\gamma(x)}$. All these jets form a bundle

$$\lambda J^r\big((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2)\big)\,, \qquad \text{in short} \quad \lambda J^r(M_1, M_2)\,.$$

If x^i, y^p are local leaf coordinates on M_1, and u^t, w^a are local leaf coordinates on M_2, then the induced coordinates on $\lambda J^r(M_1, M_2)$ are

$$w^a_\beta\,, \quad |\beta| \leq r\,,$$

where β is a multiindex corresponding to M_1. If $(M_3, \mathcal{F}_3)$ is another foliated manifold and $j^r_{\gamma(z)}\delta \in \lambda J^r(M_2, M_3)$, we can construct the composition

$$j^r_{\gamma(z)}\delta \circ j^r_z \gamma = j^r_x(\delta \circ \gamma) \in \lambda J^r(M_1, M_3)\,. \tag{6.33}$$

For the product $(M_2 \times M_3, \mathcal{F}_2 \times \mathcal{F}_3)$, one verifies directly

$$\lambda J^r(M_1, M_2 \times M_3) = \lambda J^r(M_1, M_2) \times_{M_1} \lambda J^r(M_1, M_3)\,. \tag{6.34}$$

The category $\mathcal{FM}$ of fibered manifolds and fibered morphisms can be considered as a subcategory of $\mathcal{F}ol$. For two fibered manifolds $p_1\colon Y_1 \to B_1$

and $p_2 \colon Y_2 \to B_2$, we have

$$\lambda J^r(Y_1, Y_2) = \bigcup_{(x_1,x_2)\in B_1\times B_2} J^r(Y_{1x_1}, Y_{2x_2})\,.$$

Every $\mathcal{F}ol$-morphism $f \colon M_1 \to M_2$ defines a section

$$\lambda^r f \colon M_1 \to \lambda J^r(M_1, M_2)\,, \quad (\lambda^r f)(z) = j^r_z(f|L_z)\,.$$

We say that $j^k_z(\lambda^r f)$ is the leafwise (k, r)-jet of f at z.
We write

$$J^k\lambda J^r(M_1, M_2) = J^k\big(\lambda J^r(M_1, M_2) \to M_1\big) \tag{6.35}$$

for the bundle of all these jets. The induced coordinates on (6.35) are

$$v^t_\alpha\,, \quad |\alpha| \le k\,, \quad w^a_{\alpha,\beta}\,, \quad |\alpha| + |\beta| \le k + r\,,$$

where α is the multiindex corresponding to x^i. For $l < k$, $j^l_z\lambda^{r+k-l}f$ depends on $j^r_x\lambda^r f$ only. This defines a projection

$$\pi^{k,r}_l \colon J^k\lambda J^r(M_1, M_2) \to J^l\lambda J^{r+k-l}(M_1, M_2)\,, \quad l < k\,.$$

Lemma. *Let $g \colon M_2 \to M_3$ be another $\mathcal{F}ol$-morphism. Then the value $j^k_z(\lambda^r(g \circ f))$ depends on $j^k_z(\lambda^r f)$ and $j^k_{f(z)}(\lambda^r g)$ only.*

Proof. Consider the coordinate expression of f

$$v^t = f^t(x^i)\,, \quad w^a = f^a(x^i, y^p) \tag{6.36}$$

and the analogous coordinate expression of g. Then our assertion follows directly from (6.33). □

This defines the composition of leafwise (k, r)-jets.
Consider a $(p + 1)$-tuple $\varrho = (r_0, r_1, \dots, r_p)$ of integers such that

$$r_0 > r_1 > \cdots > r_p \ge 0\,.$$

For f, $g \in \mathcal{F}ol(M_1, M_2)$, we define $j^\varrho_z f = j^\varrho_z g$, $z \in M_1$, by

$$\lambda^{r_0} f(z) = \lambda^{r_0} g(z)\,, \quad j^i_z\lambda^{r_i} f = j^i_z\lambda^{r_i} g\,, \quad i = 1, \dots, p$$

and we write $J^\varrho(M_1, M_2)$ for the set of all these jets. Clearly, $J^\varrho(M_1, M_2)$ is a fibered manifold over $M_1 \times M_2$. Indeed, we start with arbitrary quantities v^t_α, $|\alpha| \le p$ and $w^a_{\alpha_p,\beta_p}$, $|\alpha_p| \le p$, $|\alpha_p| + |\beta_p| \le p + r_p$. Since $r_{p-1} > r_p$, we can prescribe arbitrarily the remaining $w_{\alpha_{p-1},\beta_{p-1}}$, $|\alpha_{p-1}| \le p - 1$,

$|\alpha_{p-1}| + |\beta_{p-1}| \leq p - 1 + r_{p-1}$. In the last step of this recurrence procedure we have w_{α_1,β_1}, $|\alpha_1| \leq 1$, $|\alpha_1| + |\beta_1| \leq 1 + r_1$. Since $r_0 > r_1$, we may prescribe the remaining $w^a_{\beta_0}$, $|\beta_0| \leq r_0$ arbitrarily. So we obtain the induced coordinates on $J^\varrho(M_1, M_2)$.
By (6.34), we deduce

$$J^\varrho(M_1, M_2 \times M_3) = J^\varrho(M_1, M_2) \times_{M_1} J^\varrho(M_1, M_3) . \tag{6.37}$$

We say that two $\mathcal{F}ol$-morphisms f, $g\colon M_1 \to M_2$ determine the same transversal r-jet $\tau^r_z f = \tau^r_z g$ at $z \in M_1$, if

$$j^r_z(h \circ f) = j^r_z(h \circ g)$$

for every local function h on M_2 constant on the leaves. We write $\tau J^r(M_1, M_2)$ for the bundle of these jets. Clearly,

$$\tau J^r(M_1, M_2 \times M_3) = \tau J^r(M_1, M_2) \times_{M_1} \tau J^r(M_1, M_3) .$$

Since the classical definition of $j^r_x f = j^r_x g$ for two smooth maps f, $g\colon N_1 \to N_2$, $x \in N_1$, is equivalent to $j^r_x(h \circ f) = j^r_x(h \circ g)$ for every smooth function h on N_2, we have a canonical projection

$$\lambda J^r(M_1, M_2) \to \tau J^r(M_1, M_2) .$$

More generally, we have a projection

$$J^k \lambda J^r(M_1, M_2) \to \tau J^k(M_1, M_2) , \quad j^k_z(\lambda^r f) \mapsto \tau^k_z f . \tag{6.38}$$

In particular, (6.38) defines the composition of transversal r-jets. For an $\mathcal{FM}$-morphism $f\colon Y_1 \to Y_2$ over $\underline{f}\colon M_1 \to M_2$, $\tau^r_z f$ coincides with $j^r_{p_1(z)} \underline{f}$.

We point out that there is another characterization of $j^r_z \lambda^r f$. We have $j^k f\colon M_1 \to J^k(M_1, M_2)$.

Proposition. *The leafwise (k, r)-jets $j^r_z \lambda^r f$ are in bijection with r-jets $j^r_z(j^k f \mid L_z)$.*

Proof. If (6.36) is the coordinate expression of f, then $\lambda^r f$ is expressed by $v^t(x^i)$, $D_\beta w^a(x^i, y^p)$, $|\beta| \leq r$. Hence $j^k_x \lambda^r f$ is determined by

$$D_\alpha v^t, D_{\alpha_1,\beta_1}(D_\beta w^a) , \quad |\alpha| \leq k_1, \ |\alpha_1| + |\beta_1| \leq k .$$

On the other hand, the coordinate expression of $j^k f$ is

$$D_\alpha v^t, \ |\alpha| \lneq k \quad \text{and} \quad D_{\alpha_1,\beta_1} w^a, \quad |\alpha_1| + |\beta_1| \leq k ,$$

so that $j^r_z(j^k f \mid L_z)$ is further determined by $D_\beta(D_{\alpha_1,\beta_1} w^a)$, $|\beta| \leq r$.
By (6.34), we have a projection $J^\varrho(M_1, M_2) \to \tau J^p(M_1, M_2)$. For every $q \geq p$, we define

$$J^{\varrho,q}(M_1, M_2) = J^\varrho(M_1, M_2) \times_{\tau J^p(M_1,M_2)} \tau J^q(M_1, M_2)\,.$$

Write $G\mathcal{F}ol(M_1, M_2)$ for the set of all germs of foliation respecting maps of M_1 into M_2. The rule

$$j^{\varrho,q}_{M_1,M_2}(\mathrm{germ}_z\, f) = (j^\varrho_z f, \tau^q_z f) =: j^{\varrho,q}_z f$$

is a surjective map $G\mathcal{F}ol(M_1, M_2) \to J^{\varrho,q}(M_1, M_2)$. Analogously to (6.37) we have

$$J^{\varrho,q}(M_1, M_2 \times M_3) = J^{\varrho,q}(M_1, M_2) \times_{M_1} J^{\varrho,q}(M_1, M_3)\,.$$

□

Now we apply an abstract viewpoint similar to Section 6.9. Consider a rule F transforming every pair $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ of foliated manifolds into a fibered manifold $F((M_1, \mathcal{F}_1), (M_2, \mathcal{F}_2))$ (in short: $F(M_1, M_2)$) over $M_1 \times M_2$ and a system of maps $\varphi_{M_1,M_2}\colon G\mathcal{F}ol(M_1, M_2) \to F(M_1, M_2)$ commuting with the projection $G\mathcal{F}ol(M_1, M_2) \to M_1 \times M_2$ and $F(M_1, M_2) \to M_1 \times M_2$ for all M_1, M_2. Analogously to Section 6.9, we formulate the following requirements I–IV.
I. Every $\varphi_{M_1,M_2}\colon G\mathcal{F}ol(M_1, M_2) \to F(M_1, M_2)$ is surjective.
II. If $B_1, \overline{B}_1 \in G_{z_1}\mathcal{F}ol(M_1, M_2)_{z_2}$ and $B_2, \overline{B}_2 \in G_{z_2}\mathcal{F}ol(M_2, M_3)_{z_3}$ satisfy $\varphi(B_1) = \varphi(\overline{B}_1)$ and $\varphi(B_2) = \varphi(\overline{B}_2)$, then $\varphi(B_2 \circ B_1) = \varphi(\overline{B}_2 \circ \overline{B}_1)$.
By I and II, we have a well defined composition (denoted by the same symbol as the composition of germs and maps)

$$X_2 \circ X_1 = \varphi(B_2 \circ B_1)$$

for $X_1 = \varphi(B_1) \in F_{z_1}(M_1, M_2)_{z_2}$ and $X_2 = \varphi(B_2) \in F_{z_2}(M_2, M_3)_{z_3}$. Write $\varphi_z f$ for $\varphi(\mathrm{germ}_z\, f)$. For another pair $\overline{M}_1$, $\overline{M}_2$ of foliated manifolds, every local $\mathcal{F}ol$-isomorphism $f\colon M_1 \to \overline{M}_1$ and every $\mathcal{F}ol$-morphism $g\colon M_2 \to \overline{M}_2$ induce a map $F(f, g)\colon F(M_1, M_2) \to F(\overline{M}_1, \overline{M}_2)$ by

$$F(f, g)(X) = (\varphi_{z_2} g) \circ X \circ \varphi_{f(z_1)}(f^{-1})\,, \quad X \in F_{z_1}(M_1, M_2)_{z_2}\,,$$

where f^{-1} is constructed locally. We require
III. (Regularity) Each map $F(f, g)$ is smooth.
IV. (Product property) $F(M_1, M_2 \times M_3) = F(M_1, M_2) \times_{M_1} F(M_1, M_3)$.

Definition. A pair (F, φ) satisfying I–IV will be called a jet-like homomorphism on germs of foliation respecting maps.

Clearly, $(J^{\varrho,q}, j^{\varrho,q})$ is a jet-like homomorphism, provided the composition is defined componentwise. The following result is an analogy of Theorem 6.9.

Theorem. *Every jet-like homomorphism on germs of foliation respecting maps is of the form* $(J^{\varrho,q}, j^{\varrho,q})$.

The proof, that uses heavily the technique of Weil algebras, can be found in [DoKoMi] or [Ko16].

Chapter 7

Nonholonomic and semiholonomic jets

7.1 The skeletons

The classical, or holonomic, r-jets $X = j^r_x\varphi$ of smooth maps $\varphi\colon M \to N$ form a fibered manifold $J^r(M,N) \to M \times N$ with respect to the source and target projections $\alpha X = x \in M$ and $\beta X = \varphi(x) \in N$. All r-jets form a category J^r over pointed manifolds (M,x): if $X \in J^r_x(M,N)_y$ and $Z = j^r_y\psi \in J^r_y(N,Q)_z$, then $Z \circ X = j^r_x(\psi \circ \varphi) \in J^r_x(M,Q)_z$. We write $L^r_{m,n} = J^r_0(\mathbb{R}^m, \mathbb{R}^n)_0$. Then

$$L^r = \bigcup_{m,n\in\mathbb{N}} L^r_{m,n} \tag{7.1}$$

is a category over integers called the skeleton of J^r. Clearly, J^r can be reconstructed from L^r: if $\dim M = m$ and $\dim N = n$, then $J^r(M,N)$ coincides with the associated bundle

$$J^r(M,N) = (P^rM \times P^rN)\big[L^r_{m,n}\big]\,. \tag{7.2}$$

The bundle $\widetilde{J}^2(M,N)$ of nonholonomic 2-jets of M into N is the space of 1-jets $X = j^1_x f$ of the α-sections of $J^1(M,N)$, i.e. the maps $f\colon M \to J^1(M,N)$ satisfying $\alpha f(u) = u$, $u \in M$. This is a bundle over $M \times N$ with respect to the source projection $\alpha(j^1_x f) = x$ and the target projection $\beta(j^1_x f) = \beta(f(x)) \in N$, where β on the right hand side is the target projection of $J^1(M,N)$. Local coordinates x^i on M and y^p on N induce the additional coordinates y^p_i on $J^1(M,N)$. It will be useful to write $y^p = y^p_0$.

So the coordinate expression of an α-section $f(u)$ is $f_0^p(u)$, $f_i^p(u)$. Then x^i and $y^p_{h_1h_2}$, $h_1, h_2 = 0, 1, \dots, m$,

$$y^p_{00} = f^p_0(x),\ y^p_{i0} = f^p_i(x),\ y^p_{0i} = \frac{\partial f^p_0(x)}{\partial u^i},\ y^p_{ij} = \frac{\partial f^p_i(x)}{\partial u^j} \tag{7.3}$$

are the induced coordinates on $\widetilde{J}^2(M,N)$. The subset $J^2(M,N) \subset \widetilde{J}^2(M,N)$ is characterized by

$$y^p_{i0} = y^p_{0i} \quad \text{and} \quad y^p_{ij} = y^p_{ji}\,. \tag{7.4}$$

We have two projections $\varrho_1,\ \varrho_2\colon \widetilde{J}^2(M,N) \to J^1(M,N)$, $\varrho_1(j^1_x f) = f(x)$, $\varrho_2(j^1_x f) = j^1_x(\beta \circ f)$. In coordinates, $\varrho_1(X) = (x^i, y^p_{00}, y^p_{i0})$, $\varrho_2(X) = (x^i, y^p_{00}, y^p_{0i})$. We say that $X \in \widetilde{J}^2_x(M,N)_y$ is semiholonomic, if $\varrho_1(X) = \varrho_2(X)$, i.e. $y^p_{i0} = y^p_{0i}$. We write $\overline{J}^2(M,N)$ for the bundle of all semiholonomic 2-jets of M into N. By (7.4), $J^2(M,N) \subset \overline{J}^2(M,N)$.
Even the nonholonomic 2-jets form a category $\widetilde{J}^2$ over pointed manifolds. For $X = j^1_x f(u) \in \widetilde{J}^2_x(M,N)_y$ and $Z = j^1_y g(v) \in \widetilde{J}^2_y(N,Q)_z$ one defines

$$Z \circ X = j^1_x\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}^2_x(M,Q)_z \tag{7.5}$$

with the composition of 1-jets on the right hand side. If X and Z are holonomic, then (7.5) coincides with the composition of 2-jets. Indeed, if $X = j^1_x(j^1_u\varphi)$ and $Z = j^1_y(j^1_u\psi)$, then

$$j^1_x\big(j^1_u(\psi \circ \varphi)\big) = j^1_x\big(j^1_{\varphi(u)}\psi \circ j^1_u\varphi\big) \tag{7.6}$$

and the right hand sides of (7.5) and (7.6) are the same.
We say that $\widetilde{L}^2_{m,n} = \widetilde{J}^2_0(\mathbb{R}^m, \mathbb{R}^n)_0$ is the skeleton of $\widetilde{J}^2$ and we have

$$\widetilde{J}^2(M,N) = (P^2M \times P^2N)[\widetilde{L}^2_{m,n}]\,. \tag{7.7}$$

7.2 Nonholonomic r-jet categories

Assume by induction that we have constructed the nonholonomic $(r-1)$-jet bundle $\widetilde{J}^{r-1}(M,N) \to M \times N$ with the source projection $\alpha\colon \widetilde{J}^{r-1}(M,N) \to M$ and the target projection $\beta\colon \widetilde{J}^{r-1}(M,N) \to N$ and $r-1$ canonical projections $\sigma_1, \dots \sigma_{r-1}\colon \widetilde{J}^{r-1}(M,N) \to \widetilde{J}^{r-2}(M,N)$. Then we define the bundle of nonholonomic r-jets of M into N to be the space $\widetilde{J}^r(M,N)$ of 1-jets of the α-sections $f\colon M \to \widetilde{J}^{r-1}(M,N)$. This is a bundle over $M \times N$ with respect to the source and target projections $\alpha(j^1_x f) = x$ and

$\beta(j^1_x f) = \beta(f(x)) \in N$, where β on the right hand side means the target projection of $\widetilde{J}^{r-1}(M,N)$.
Assume by induction that $(x^i, y^p_{h_1\dots h_{r-1}})$, $h_1,\dots,h_{r-1} = 0,1,\dots,m$ are the local coordinates on $\widetilde{J}^{r-1}(M,N)$ induced by x^i on M and y^p on N and the coordinate form of σ_s is

$$\sigma_s(x^i, y^p_{h_1\dots h_{r-1}}) = (x^i, y^p_{h_1\dots h_{s-1}0h_{s+1}\dots h_{r-1}})\,, \quad s = 1,\dots r-1\,.$$

The coordinate expression of f is $f^p_{h_1\dots h_{r-1}}(u)$. This induces the coordinates $(x^i, y^p_{h_1\dots h_r})$, $h_1,\dots,h_r = 0,1,\dots,m$ on $\widetilde{J}^r(M,N)$ by

$$y^p_{h_1\dots h_{r-1}0} = f^p_{h_1\dots h_{r-1}}(x),\ y^p_{h_1\dots h_{r-1}j} = \frac{\partial f^p_{h_1\dots h_{r-1}}(x)}{\partial u^j}\,, \quad j = 1,\dots,m\,. \tag{7.8}$$

We construct r projections $\varrho_s \colon \widetilde{J}^r(M,N) \to \widetilde{J}^{r-1}(M,N)$ by

$$\varrho_1(j^1_x f) = f(x),\dots,\varrho_{s+1}(j^1_x f) = j^1_x(\sigma_s \circ f)\,, \quad s = 1,\dots,r-1\,. \tag{7.9}$$

Hence the coordinate form of σ_s is

$$\sigma_s(x^i, y^p_{h_1\dots h_r}) = (x^i, y^p_{h_1\dots h_s 0 h_{s+2}\dots h_r})\,. \tag{7.10}$$

The composition of $X = j^1_x f \in \widetilde{J}^r(M,N)_y$ and $Z = j^1_y g \in \widetilde{J}^r(N,Q)_z$ is defined by

$$Z \circ X = j^1_x\big(g(\beta f(u)) \circ f(u)\big) \in \widetilde{J}^r_x(M,Q)_z \tag{7.11}$$

with the composition of nonholonomic $(r-1)$-jets on the right hand side. It is easy to deduce that (7.11) is associative. So $\widetilde{J}^r$ is a category over pointed manifolds. Modifying (7.6), we deduce that for two holonomic jets, (7.11) coincides with the classical composition.

Lemma. *Every projection ϱ_s, $s = 1,\dots,r$, preserves the composition of nonholonomic jets.*

Proof. For ϱ_1 this follows directly from (7.11). Assume by induction that every σ_s, $s = 1,\dots,r-1$ preserves the composition of nonholonomic jets. Then we have

$$\begin{aligned}\varrho_s\big(j^1_x g(\beta f(u)) \circ f(u)\big) &= j^1_x\big(\sigma_{s-1}(g(\beta f(u) \circ f(u))\big)\\ &= j^1_x\big(\varrho_{s-1}\big(g(\beta f(u)) \circ \varrho_{s-1}(f(u)\big)\\ &= \sigma_s(j^1_y g) \circ \sigma_s(j^1_x f)\,.\end{aligned} \tag{7.12}$$

□

Remark. This description of nonholonomic r-jets was developed by G. Virsik, [Vi].

An element $X \in \widetilde{J}^r(M,N)$ is called semiholonomic r-jet, if $\varrho_1 X = \dots = \varrho_r X$. The bundle of all semiholonomic r-jets is denoted by $\overline{J}^r(M,N)$. Write $\langle h_1,\dots,h_r\rangle = (i_1,\dots,i_s)$ for the subsequence of all nonzero indices and $|h_1\dots h_r|$ for the set $\{i_1,\dots,i_s\}$. Then the elements of $\overline{J}^r(M,N)$ are characterized by

$$y^p_{h_1\dots h_r} = y^p_{l_1\dots l_r} \quad \text{whenever} \quad \langle h_1\dots h_r\rangle = \langle l_1\dots l_r\rangle\,.$$

Hence no zero subscripts are needed in the semiholonomic case. The inclusion $J^r(M,N) \hookrightarrow \widetilde{J}^r(M,N)$ is defined by

$$j^r_x\varphi \mapsto j^1_x(j^{r-1}\varphi)\,, \quad \varphi\colon M\to N\,. \tag{7.13}$$

The elements of $J^r(M,N)$ are characterized by

$$y^p_{h_1\dots h_r} = y^p_{l_1\dots l_r} \quad \text{whenever} \quad |h_1\dots h_r| = |l_1\dots l_r|\,.$$

Hence $J^r(M,N) \subset \overline{J}^r(M,N)$.
The compositions of various canonical projections $\widetilde{J}^r(M,N) \to \widetilde{J}^{r-1}(M,N)$ define $\binom{r}{k}$ canonical projections $\widetilde{J}^r(M,N) \to \widetilde{J}^k(M,N)$. They are in bijection with the fixations of $(r-k)$ zeros in the sequence $h_1,\dots,h_r$. Given a fixation $\varkappa$ of $r-k$ elements from $1,\dots,r$, the coordinate form of the corresponding projection is $(x^i, y^p_{h_1\dots h_r}) \mapsto (x^i, y^p_{\varkappa(h_1,\dots,h_r)})$, where $\varkappa(h_1,\dots,h_r)$ means that we replace the indices at the prescribed places by zeros. Then we can define an element of $\widetilde{J}^r(M,N)$ to be k-semiholonomic, if all its canonical projections into $\widetilde{J}^k(M,N)$ coincide. We write $\widetilde{J}^{r,k}(M,N)$ for the bundle of all k-semiholonomic r-jets. Clearly, $(r-1)$-semiholonomic means semiholonomic in the classical sense.
Hence, Lemma 7.2 implies that the composition of two k-semiholonomic r-jets is a k-semiholonomic r-jet. In particular, the composition of two classical semiholonomic r-jets is a classical semiholonomic r-jet.
Even $\widetilde{J}^r$ can be interpreted as a functor on $\mathcal{M}f_m \times \mathcal{M}f$ by

$$\widetilde{J}^r(f,g)(X) = (j^r_y g)\circ X \circ (j^r_x f)^{-1} \tag{7.14}$$

with the composition of nonholonomic r-jets. Using induction, one verifies directly

$$\widetilde{J}^r(M, N_1\times N_2) = \widetilde{J}^r(M,N_1)\times_M \widetilde{J}^r(M,N_2)\,. \tag{7.15}$$

The same holds in the k-semiholonomic case.

Write

$$\widetilde{L}^r_{m,n} = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^n)_0\,, \quad \widetilde{L}^r = \bigcup_{m,n\in\mathbb{N}} \widetilde{L}^r_{m,n}\,. \tag{7.16}$$

Then $\widetilde{L}^r$ is a category over integers, called the skeleton of $\widetilde{J}^r$. Analogously to the holonomic case,

$$\widetilde{J}^r(M,N) = (P^rM \times P^rN)[\widetilde{L}^r_{m,n}]\,. \tag{7.17}$$

For the semiholonomic r-jets we write

$$\overline{L}^r_{m,n} = \overline{J}^r_0(M,N)_0\,, \quad \overline{L}^r = \bigcup_{m,n\in\mathbb{N}} \overline{L}^r_{m,n} \tag{7.18}$$

and we have $\overline{J}^r(M,N) = (P^rM \times P^rN)[\overline{L}^r_{m,n}]$.
We say that $X \in \widetilde{J}^r_x(M,N)_y$ is regular, if there exists $Z \in \widetilde{J}^r_y(N,M)_x$ such that $Z \circ X = j^r_x \,\mathrm{id}_M$. The general concept of r-jet category can be introduced as follows.

Definition. A nonholonomic r-jet category C is a rule transforming every pair (M,N) of manifolds into a fibered submanifold $C(M,N) \subset \widetilde{J}^r(M,N)$ such that

(i) $J^r(M,N) \subset C(M,N)$ is a fibered submanifold,

(ii) if $X \in C_x(M,N)_y$ and $Z \in C_y(N,Q)_z$, then $Z \circ X \in C_x(M,Q)_z$,

(iii) if $X \in C_x(M,N)_y$ is regular in $\widetilde{J}^r$, then there exists $Z \in C_y(N,M)_x$ such that $Z \circ X = j^r_x \,\mathrm{id}_M$,

(iv) $C(M, N_1 \times N_2) = C(M,N_1) \times_M C(M,N_2)$.

Write $L^C_{m,n} = C_0(\mathbb{R}^m, \mathbb{R}^n)_0$ and $L^C = \bigcup_{m,n\in\mathbb{N}} L^C_{m,n}$. This skeleton is a subcategory of $\widetilde{L}^r_{m,n}$ and we have

$$C(M,N) = (P^rM \times P^rN)[L^C_{m,n}]\,. \tag{7.19}$$

If we consider a fibered manifold $Y \to M$, the local fiber coordinates on Y define a local identification of Y with $M \times N$. Then we can apply the previous ideas of this section.

7.3 Nonholonomic $[r,s]$-jets

In a certain sense these jets represent the fundamental example of nonholonomic jets. Consider the r-th jet prolongation J^rY of a fibered manifold $p\colon Y \to M$. The nonholonomic $[r,s]$-jet prolongation of Y is defined by

$$J^{r,s}Y = J^s(J^rY \to M)\,.$$

If Y is the product $M \times N \to M$, the elements of $J^{r,s}(M \times N \to M) =: J^{r,s}(M,N)$ are called nonholonomic $[r,s]$-jets of M into N. The canonical injection $J^{r+s}Y \hookrightarrow J^{r,s}Y$ is of the form

$$j^{r+s}_x\sigma \mapsto j^s_x(j^r\sigma)\,. \tag{7.20}$$

In particular, $J^{r+s}(M,N) \hookrightarrow J^{r,s}(M,N)$. For $s=0$, we have $J^{r,0}(M,N) = J^r(M,N)$.
Write $\beta_1\colon J^{r,s}Y \to J^rY$ for the target jet projection. The target projection $\beta\colon J^rY \to Y$ is extended into a map $\beta_2 := J^s\beta\colon J^{r+s}Y \to J^sY$.

Lemma. *$X \in J^{r,s}_x(M,N)_y$ is invertible, iff both $\beta_1 X \in J^r_x(M,N)_y$ and $\beta_2 X \in J^s_x(M,N)_y$ are invertible.*

Proof. We have $X = j^s_x F(u)$, $F\colon M \to J^r(M,N)$. Since $\beta_1 X = F(x)$ is invertible, we can locally construct $F^{-1}(u)$. Since $\beta_2 X = j^s_x f$ is invertible, there exists locally the inverse map $\widetilde{f}$ of f. Then

$$\widetilde{X} := j^s_y(F^{-1} \circ \widetilde{f}) \in J^{r,s}_y(N,M))_x$$

satisfies $\widetilde{X} \circ X = E^{r+s}_{x,M}$ and $X \circ \widetilde{X} = E^{r+s}_{y,N}$, where $E^{r+s}_{x,M} = j^{r+s}_x \operatorname{id}_M$ and $E^{r+s}_{y,N} = j^{r+s}_y \operatorname{id}_N$. □

An element $X \in J^{r,s}_x(M,N)$ is called regular, if there exists $Z \in J^{r,s}_y(N,M)$ such that $Z \circ X = E^{r+s}_{x,M}$. One verifies directly that

$$X \text{ is regular, iff both } \beta_1 X \text{ and } \beta_2 X \text{ are regular.} \tag{7.21}$$

We define

$$L^{r,s}_{m,n} = J^{r,s}_0(\mathbb{R}^m,\mathbb{R}^n)_0 \quad \text{and} \quad L^{r,s} = \bigcup_{m,n\in\mathbb{N}} L^{r,s}_{m,n}\,. \tag{7.22}$$

The following assertion describes $L^{r,s}$ in terms of L^r.

Proposition. *We have* $L^{r,s}_{m,n} = L^s_{m,n} \times T^s_m L^r_{m,n}$ *with the composition*

$$Z \circ X = \big(Z_1 \circ X_1, T^s_m \varkappa^r_{m,n,p}(Z_2 \circ X_1, X_2)\big) \tag{7.23}$$

where $\varkappa^r_{m,n,p}\colon L^r_{n,p} \times L^r_{m,n} \to L^r_{m,p}$ *is the composition in* L^r, $X = (X_1, X_2) \in L^{r,s}_{m,n}$ *and* $Z = (Z_1, Z_2) \in L^{r,s}_{n,p}$.

Proof. Consider the canonical identification $J^r(\mathbb{R}^m, \mathbb{R}^n) = \mathbb{R}^m \times L^r_{m,n} \times \mathbb{R}^n$ defined by the translations on $\mathbb{R}^m$ and $\mathbb{R}^n$. Hence a section $F\colon \mathbb{R}^m \to J^r(\mathbb{R}^m, \mathbb{R}^n)$ is identified with a pair of maps $f_1\colon \mathbb{R}^m \to \mathbb{R}^n$, $f_2\colon \mathbb{R}^m \to L^r_{m,n}$, so that $X = j^s_0 F$ is identifies with $(j^s_0 f_1, j^s_0 f_2) \in L^s_{m,n} \times T^s_m L^r_{m,n}$. In this special case, (7.11) implies (7.23). □

Further, consider $X \in \widetilde{J}^r(M, N)$ and $\sigma_1 X, \dots, \sigma_r X \in \widetilde{J}^{r-1}(M, N)$ with σ_i from (7.10). Analogously to the above lemma, we obtain X is invertible, iff all $\sigma_1 X, \dots, \sigma_r X$ are invertible. Moreover,

$$X \text{ is regular, iff all } \sigma_1 X, \dots, \sigma_r X \text{ are regular.}$$

7.4 Semiholonomic r-jets

A direct way how to introduce the bundle $\overline{J}^r(M, N)$ of semiholonomic r-jets together with the projection $\pi^r_{r-1}\colon \overline{J}^r(M, N) \to \overline{J}^{r-1}(M, N)$ is the following induction. The elements of $\overline{J}^r(M, N)$ are of the form $j^1_x s$, where $s\colon M \to \overline{J}^{r-1}(M, N)$ is a section satisfying

$$s(x) = j^1_x(\pi^{r-1}_{r-2} \circ s)\,. \tag{7.24}$$

In the case of a fibered manifold $p\colon Y \to M$, the elements of $\overline{J}^r Y$ are of the form $j^1_x s$, where s is a section of $\overline{J}^{r-1} Y$ satisfying

$$s(x) = j^1_x(p^{r-1}_{r-2} \circ s)\,, \tag{7.25}$$

where $p^{r-1}_{r-2}\colon \overline{J}^{r-1} Y \to \overline{J}^{r-2} Y$ is defined in the induction procedure.
In the holonomic case, Proposition 12.11 from [KMS] asserts that
(a) $J^r(M, N) \to J^{r-1}(M, N)$ is an affine bundle, whose associated vector bundle is the pullback of $TN \otimes S^r T^* M$ over $J^{r-1}(M, N)$; this implies directly an analogous result for the case of a fibered manifold $p\colon Y \to M$ with the induced projection $J^r Y \to J^{r-1} Y$,
(b) $J^r Y \to J^{r-1} Y$ is an affine bundle whose associated vector bundle is the pullback of $VY \otimes S^r T^* M$ over $J^{r-1} Y$.

In the semiholonomic case, we have canonical projections

$$\pi^r_{r-1}\colon \overline{J}^r(M,N) \to \overline{J}^{r-1}(M,N) \quad \text{and} \quad p^r_{r-1}\colon \overline{J}^r Y \to \overline{J}^{r-1} Y\,. \tag{7.26}$$

Technically it is better to start with the fibered case.

Theorem. $p^r_{r-1}\colon \overline{J}^r Y \to \overline{J}^{r-1} Y$ *is an affine bundle, whose associated vector bundle is the pullback of* $VY \otimes \bigotimes^r T^*M$ *over* $\overline{J}^{r-1}Y$.

Proof. (by P. Libermann, [Li69]) We proceed by induction. The case $r = 1$ is well known. Let $v \in \overline{J}^r Y$, $v_1 = p^r_{r-1}(v)$, $y = p^r_0(v)$, $x = p(y)$. By the induction hypothesis, the kernel of the projection p^{r-1}_{r-2} at v_1 is $V_yY \otimes \bigotimes^{r-1} T^*M$. For every $v' \in J^1\overline{J}^{r-1}Y$ such that $\beta(v') = \beta(v) = v_1$, the jets v and v', that can be considered as linear maps $T_xM \to T_{v_1}\overline{J}^{r-1}Y$, satisfy $v - v' \in \mathrm{Lin}\,(T_xM, V_yY \otimes \bigotimes^{r-1} T^*_xM)$, i.e., $v - v' \in V_yY \otimes \bigotimes^{r} T^*_xM$. □

In the manifold case, the above Proposition implies easily

Corollary. $\pi^r_{r-1}\colon \overline{J}^r(M,N) \to \overline{J}^{r-1}(M,N)$ *is an affine bundle, whose associated vector bundle is the pullback of* $TN \otimes \bigotimes^r T^*M$ *over* $\overline{J}^{r-1}(M,N)$.

For local evaluations, it is sometimes useful to consider the product $V \times W$ of two vector spaces. Then the previous results are specified to

$$J^r(V,W) = V \times W \times \Big(\sum_{i=1}^{r} W \otimes S^r V^*\Big)\,, \tag{7.27}$$

$$\overline{J}^r(V,W) = V \times W \times \Big(\sum_{i=1}^{r} W \otimes \bigotimes^r V^*\Big)\,. \tag{7.28}$$

7.5 The difference tensor of a semiholonomic 2-jet

Consider a fibered manifold $p\colon Y \to M$ with the fiber coordinates x^i, y^p. We know that $J^1Y \to Y$ is an affine bundle with the associated vector bundle $VY \otimes T^*M$. Hence $\varrho_1\colon \widetilde{J}^2Y \to J^1Y$ is also an affine bundle with the associated vector bundle $VJ^1Y \otimes T^*M$. In particular, $\overline{J}^2Y \to J^1Y$ is an affine bundle with the associated vector bundle $VY \otimes \otimes^2 T^*M$ with coordinates

$$x^i, y^p = y^p_{00}\,, \quad y^p_i = y^p_{i0} = y^p_{0i}\,, \quad y^p_{ij}\,.$$

In the product case $Y = M \times N$, consider a system of linear maps $\varphi(u)\colon T_uM \to TN$, $\eta^p = y^p_i(u)\xi^i$. This yields a local map $TM \to TN$, whose tangent map at each point of T_xM is determined by $j^1_x\varphi$. So the nonholonomic 2-jet $Z = j^1_x\varphi(u) \in \overline{J}^2_x(M,N)_y$ can be interpreted as a map $(TTM)_x \to (TTN)_y$ of the form

$$\eta^p = y^p_i\xi^i\,, \quad dy^p = y^p_{0i}dx^i\,, \quad d\eta^p = y^p_{ij}\xi^i dx^j + y^p_i d\xi^i\,. \tag{7.29}$$

Consider the canonical involution ι of the iterated tangent functor, $\iota_M(\xi^i, dx^i, d\xi^i) = (dx^i, \xi^i, d\xi^i)$ and $Z \in \overline{J}^2_x(M,N)_y$ of the form (7.29) with $y^p_{0i} = y^p_i$. Then the coordinate expression of $\iota_N \circ Z \circ \iota_M$ is

$$\eta^p = y^p_i\xi^i\,,\ dy^p = y^p_i dx^i\,,\ d\eta^p = y^p_{ji}\xi^i dx^j + y^p_i dx^i\,. \tag{7.30}$$

This map corresponds to another semiholonomic 2-jet

$$\varkappa(Z) \in \overline{J}^2_x(M,N)_y\,,\ \varkappa(y^p_i, y^p_{ij}) = (y^p_i, y^p_{ji})\,. \tag{7.31}$$

Definition. The map $\varkappa$ is called the canonical involution of semiholonomic 2-jets.

Since $\overline{J}^2(M,N) \to J^1(M,N)$ is an affine bundle, Z and $\varkappa(Z)$ determine a tensor

$$\Delta(Z) = Z - \varkappa(Z) \in T_yN \otimes \Lambda^2 T^*_x M \tag{7.32}$$

called the difference tensor of semiholonomic 2-jet Z. (J. Pradines uses the name “dissymétrie”, [Pr].) The coordinate expression of $\Delta(Z)$ is

$$x^i, y^p, y^p_{[ij]}\,. \tag{7.33}$$

Clearly, Z is holonomic, if and only if $\Delta(Z) = 0$.

Example. We present the first remarkable application of this concept. Consider a general connection $\Gamma\colon Y \to J^1Y$ on an arbitrary fibered manifold $Y \to M$. If Γ is viewed as a morphism over M, we can construct $J^1\Gamma\colon J^1Y \to \widetilde{J}^2Y$. Clearly, the values of the composition $\Gamma' = J^1\Gamma \circ \Gamma$ are in $\overline{J}^2Y$. The difference tensor

$$\Delta \circ \Gamma'\colon Y \to VY \otimes \Lambda^2 T^*M \tag{7.34}$$

coincides with the curvature of Γ, see (3.43).

7.6 Principal connections on P^1M

Consider a principal bundle $P(M,G)$ and write $\varrho\colon P\times G\to P$ for the right action of the structure group G on P. It was remarked in Section 3.14 that a general connection Γ on P is called principal, if it is right-invariant, i.e.,

$$\Gamma(ug)=\Gamma(u)g \tag{7.35}$$

with the induced action of G on TP on the right hand side.
We start with a few remarks on the relations of the principal connections on P^1M to the semiholonomic 2-jets. The second semiholonomic frame bundle $\overline{P}^2M$ of M is defined by

$$\overline{P}^2M=\operatorname{reg}\,\overline{J}^2_0(\mathbb{R}^m,M)\,, \tag{7.36}$$

where $X\in\operatorname{reg}\,\overline{J}^2(\mathbb{R}^m,M)$ means

$$\pi^2_1(X)\in\operatorname{reg}\,J^1_0(\mathbb{R}^m,M)=\operatorname{reg}\,T^1_mM\,.$$

This is a principal bundle with structure group $\overline{G}^2_m=\operatorname{inv}\overline{J}^2_0(\mathbb{R}^n,\mathbb{R}^m)_0$. We have a canonical injection $G^1_m\to G^2_m$, so that the inclusion $G^2_m\subset\overline{G}^2_m$ defines an injection $G^1_m\to\overline{G}^2_m$.
If x^i, x^i_j are some standard coordinates on P^1M, we write $x^i_{j,k}$ for the additional coordinates on J^1P^1M and x^i_{jk} for the additional coordinates on $\overline{P}^2M$. Consider $j^1_x\varphi\in J^1P^1M$, $\varphi(0)=j^1_0\psi$, $\psi\colon\mathbb{R}^m\to M$. Then $(j^1_0\varphi)\circ\psi\colon\mathbb{R}^m\to J^1(\mathbb{R}^m,M)$ satisfies $\beta(j^1_x\varphi\circ\psi^{-1})=j^1_x\varphi$. Define $\mu_M\colon J^1P^1M\to\overline{P}^2M$, $\mu_M((j^1_x\varphi)\circ\psi^{-1})=j^1_0((j^1_x\varphi)\circ\psi^{-1})$. The coordinate form of μ_M is

$$\mu_M(x^i,x^i_j,x^i_{j,l}\widetilde{x}^l_k)=(x^i,x^i_j,x^i_{jk})\,, \tag{7.37}$$

where $\widetilde{x}^l_k$ is the inverse matrix to x^k_l. Hence μ_M identifies J^1P^1M with $\overline{P}^2M$.
The coordinate expression of a principal connection on P^1M is

$$x^i_{j,k}=x^i_l\Gamma^l_{jk}\,. \tag{7.38}$$

It is well known that its torsion is a tensor field of type $(1,2)$ on M of the coordinate form

$$\Gamma^i_{[jk]}\,. \tag{7.39}$$

Hence (7.37) implies directly a result by S. Kobayashi, [Kob]

Proposition A. *The torsion free principal connections on P^1M are in bijection with the reductions of P^2M to $G^1_m \subset G^2_m$.*

Even the following result by P. Libermann, [Li64], follows directly from (7.37).

Proposition B. *The principal connections on P^1M are in bijection with the reductions of $\overline{P}^2M$ to $G^1_m \subset \overline{G}^2_m$.*

Remark. In higher order, the relations between connections and semi-holonomic jets were deeply studied by P. Libermann, [Li62], [Li63], [Li64], [Li65],[Li67a], [Li97]. See also a survey paper [Ko03] by the author.

7.7 Structure function of a G-structure

It is interesting that the concept of semiholonomic 2-jet plays a very useful role in the theory of G-structures. Consider a Lie subgroup $G \subset G^1_m$. A (first order) G-structure P on a manifold M is a reduction of P^1M to G. In Section 7.6, we have constructed an isomorphism $\mu_M \colon J^1P^1M \to \overline{P}^2M$. At the group level, we have an exact sequence

$$0 \to \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \to \overline{G}^2_m \to G^1_m \to 0\,. \tag{7.40}$$

Definition. The structure function $\tau(b)$ of a G-structure P at $b \in P$ is the set $\Delta(\mu(X))$ for all $X \in J^1P$, $\beta X = b$.
In coordinates, one verifies easily that our structure function coincides with the classical one, see [St].

We recall that the space $H^{0,2}(\mathfrak{g}) = \mathbb{R}^m \otimes \Lambda^2\mathbb{R}^{m*} / \operatorname{Alt}(\mathfrak{g} \otimes \mathbb{R}^{m*})$ is the Spencer cohomology class of bidegree $(0,2)$ of $\mathfrak{g}$.

Proposition. *$\tau(b)$ belongs to $H^{0,2}(\mathfrak{g})$ for every $b \in P$.*

Proof. If $Z = (\delta^i_j, z^i_{jk}) \in \overline{G}^2_m$ and $\mu(X) = (x^i, u^i_j, u^i_{jk})$, then $\mu(X) \circ Z = (x^i, u^i_j, u^i_{jk} + u^i_l z^l_{jk})$ and $\Delta(\mu(X) \circ Z) = \widetilde{u}^i_l u^l_{[jk]} + z^i_{[jk]}$. □

For every $(a^i_j) \in L^1_m$ and $A = (a^i_{jk}) \in \mathbb{R}^m \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}$, we set

$$\operatorname{ad}(a)(A) = a^i_l a^l_{mn} \widetilde{a}^m_j \widetilde{a}^n_k\,. \tag{7.41}$$

The space $\operatorname{Alt}(\mathfrak{g} \otimes \mathbb{R}^{m*})$ being invariant with respect to the action (7.41), we obtain an induced action ϱ of G on the factor spaces $H^{0,2}(\mathfrak{g})$. We have

$$\varrho(g^{-1})\tau(b) = \tau(bg)\,, \qquad g \in G,\ b \in P\,. \tag{7.42}$$

Indeed, by (7.37), if (u^i, u^i_j, u^i_{jk}) are the coordinates of $\mu(X)$, then the coordinates of X are $(u^i, u^i_j, u^i_{jl}\widetilde{u}^l_k)$. Take an element $a^i_j \in G$ and construct the image X' of X by the right translation determined by a^i_j. Then the coordinates of X' are $(u^i, u^i_k a^k_j, u^i_{ml}\widetilde{u}^l_k a^m_j)$ and the second order coordinates of $\mu(X')$ are $u^i_{lm}a^l_j a^m_k$. Hence $\Delta(\mu(X')) = \widetilde{a}^i_p \widetilde{u}^p_l u^l_{[mn]} a^m_j a^n_k$, which proves (7.42).

7.8 Prolongability and flatness

Definition. A G-structure P is called prolongable, if the intersection of $\mu(J^1P)$ and P^2M is non-empty over every element of P.

If P is prolongable, the intersection $P' = \mu(P^1M) \cap P^2M$ is said to be the prolongation of P.

Proposition. *A G-structure P is prolongable, if and only if its structure function vanishes.*

Proof. By definition, P is prolongable if and only if for every $b \in P$ there exists an $X \in J^1P$, $\beta X = b$, such that $\mu(X) \in P^2M$. This is equivalent to $\Delta(\mu(X)) = 0$, which is the same as $\tau(b) = 0$ in $H^{0,2}(\mathfrak{g})$. □

We recall that a G-structure on M is said to be flat, if it is locally isomorphic to the standard flat G-structure $\mathbb{R}^m \times G \subset P^1\mathbb{R}^m$. Hence we have

Corollary. *Every flat G-structure is prolongable.*

The converse assertion is not true in general.

Remark. In higher order, the relations between G-structures and semiholonomic jets were studied from several points of view by P. Libermann, [Li67b], [Li69], [Li73], [Li96], [Li07].

7.9 Semiholonomic 3-jet categories

A nonholonomic r-jet category C is called semiholonomic, if $C(M,N) \subset \overline{J}^r(M,N)$ for all M and N. An interesting information on the geometry of semiholonomic 3-jets is the classification of all semiholonomic 3-jets categories. It is remarkable that the technique of Weil algebras is heavily used in what follows. We denote by $\pi^r_s \colon \overline{J}^r(M,N) \to \overline{J}^s(M,N)$, $s < r$, the canonical projection. Clearly, $\overline{\mathbb{D}}^r_k = \overline{J}^r_0(\mathbb{R}^k, \mathbb{R})$ is a Weil algebra.

For $r = 3$, in the coordinates determined by (7.29), if $a = (a^p_i, a^p_{ij}, a^p_{ijk}) \in \overline{L}^3_{m,n}$ and $b = (b^\nu_p, b^\nu_{pq}, b^\nu_{pqr}) \in \overline{L}^3_{n,q}$, then $c = b \circ a = (c^\nu_i, c^\nu_{ij}, c^\nu_{ijk}) \in \overline{L}^3_{m,q}$ is of the form

$$c^\nu_i = b^\nu_p a^p_i\,, \quad c^\nu_{ij} = b^\nu_{pq} a^p_i a^q_j + b^\nu_p a^p_{ij}\,, \tag{7.43}$$
$$c^\nu_{ijk} = b^\nu_{pqr} a^p_i a^q_j a^r_k + b^\nu_{pq} a^p_{ik} a^q_j + b^\nu_{pq} a^p_i a^q_{jk} + b^\nu_{pq} a^p_{ij} a^q_k + b^\nu_p a^p_{ijk}\,.$$

Lemma. *The only subalgebra $A \subset \overline{\mathbb{D}}^3_m$ satisfying $\pi^3_2(A) = \overline{\mathbb{D}}^2_m$ is $\overline{\mathbb{D}}^3_m$.*

Proof. We prove that the kernel of the induced map $\overline{N}^3_m \to \overline{N}^2_n$ is $\overset{3}{\otimes}\mathbb{R}^{m*}$. Indeed, we deduce directly by (7.43) that the coordinate expression of the product in $\overline{\mathbb{D}}^3_m$ of x, $y \in \overline{N}^3_m$, $z = xy$, is

$$z_i = 0\,, \quad z_{ij} = x_i y_j + x_j y_i\,,$$
$$z_{ijk} = x_{ij} y_k + x_{ik} y_j + x_i y_{jk} + x_{jk} y_i + x_j y_{ik} + x_k y_{ij}\,.$$

Hence the tensor Z_{ijk} with $z_{ijk} = 1$ and all other coordinates equal to zero is obtained by multiplying $X_{ij} \in \overline{N}^3_m$ and $Y_k \in \overline{N}^3_m$, where the first and second order components of X_{ij} are $x_{ij} = 1$ and zero otherwise and the first and second order components of Y_k are $y_k = 1$ and zero otherwise. □

This lemma implies that we can restrict ourselves to the bundles $\overline{J}^{3,2}(M,N)$ of all semiholonomic 3-jets that are holonomic in the second order. In [Vo], Vosmanská deduced that all natural transformations $\overline{J}^{3,2} \to \overline{J}^{3,2}$ over the identity of J^2 form a 5-parameter family Ψ. Its coordinate expression is

$$\overline{a}^p_i = a^p_i\,,\ \overline{a}^p_{ij} = a^p_{ij} \quad \text{with} \quad a^p_{ij} = a^p_{ji}\,,$$
$$\overline{a}^p_{ijk} = a^p_{ijk} + c_1(a^p_{ikj} - a^p_{ijk}) + c_2(a^p_{jik} - a^p_{ijk})$$
$$+ c_3(a^p_{jki} - a^p_{ijk}) + c_4(a^p_{kij} - a^p_{ijk}) + c_5(a^p_{kji} - a^p_{ijk})\,.$$

We introduce $\overline{J}^{2,3}_h Y = J^1_h(J^2_h Y) \cap \overline{J}^3_h Y$ and $\overline{J}^{2,3}(M,N) = \overline{J}^{2,3}_h(M \times N \to M)$. In coordinates, $\overline{J}^{2,3}(M,N)$ is characterized by

$$a^p_{ij} = a^p_{ji}\,, \quad a^p_{ijk} = a^p_{jik}\,, \tag{7.44}$$

so that $\overline{J}^{2,3}(M,N) \subset \overline{J}^{3,2}(M,N)$. By (7.43), $\overline{J}^{2,3}$ is a semiholonomic 3-jet category. Further, for every $\psi \in \Psi$, $(\psi \circ \overline{J}^{2,3})(M,N) \subset \overline{J}^{3,2}(M,N)$ is a fibered submanifold and (7.43) implies that $\psi \circ \overline{J}^{2,3}$ is a semiholonomic 3-jet category.

If we consider an invariant tensor of degree 3 interpreted as a linear map $\iota\colon \overset{3}{\otimes}\,\mathbb{R}^{m*} \to \overset{3}{\otimes}\mathbb{R}^{n*}$ and assume it vanishes on $S^3\mathbb{R}^{m*}$, then the kernel of ι determines an invariant subspace of $V = \overset{3}{\otimes}\mathbb{R}^{m*}\big/S^3\mathbb{R}^{m*}$. By the Invariant tensor theorem, [KMS], all invariant tensors of degree 3 form a 6-parameter family

$$d_1 x_{ijk} + d_2 x_{ikj} + d_3 x_{jik} + d_4 x_{jki} + d_5 x_{kij} + d_6 x_{kji}$$

and vanishing on $S^3\mathbb{R}^{m*}$ means

$$d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = 0\,.$$

Hence we can formulate our classification result as follows.

Proposition. *All semiholonomic* 3*-jet categories are* $\overline{J}^3$, $\overline{J}^{3,2}$, J^3 *and* $\psi \circ \overline{J}^{2,3}$ *for all* $\psi \in \Psi$*.*

Remark. Using the Weil algebra technique, we classified all nonholonomic 3-jet categories in [Ko12]. The classification procedure is straightforward, but there are too many concrete cases. We refer the reader to [Ko12] for this classification.

Chapter 8

On the absolute contact differentiation

8.1 Nonholonomic and semiholonomic contact elements

In Section 2.15, we recalled the definition, by C. Ehresmann, [Eh], of a contact (n,r)-element on a manifold M as the set $Z \circ G^r_n$, where Z is a regular (n,r)-velocity on M, $n < m$. The space of all such elements is a fiber bundle $K^r_n M \to M$. Every n-submanifold $N \subset M$ defines a contact (n,r)-element $k^r_x N$ for every $X \in N$. This gives rise to a map $k^r_N \colon N \to K^r_n M$ that can be viewed as a section of the restriction $(K^r_n M)_N$ of $K^r_n M$ over N. We write $k \colon \operatorname{reg} T^r_n M \to K^r_n M$ and denote by the same symbol the induced map $k \colon \operatorname{reg} J^r(N,M) \to K^r_n M$. Hence $K^r_n M$ can be expressed as the factor space

$$K^r_n M = \operatorname{reg} T^r_n M / G^r_n\,, \quad n < m = \dim M\,. \tag{8.1}$$

Definition. The space $\widetilde{K}^r_n M$ of nonholonomic contact (n,r)-elements on M is defined by the iteration $\widetilde{K}^r_n M = K^1_n(\widetilde{K}^{r-1}_n M)$, $\widetilde{K}^1_n M = K^1_n M$.

Hence $\widetilde{K}^r_n M \to \widetilde{K}^{r-1}_n M$ is a fibered manifold. The injection $K^r_n M \hookrightarrow \widetilde{K}^r_n M$ is determined by the rule

$$k^r_x N \longmapsto k^1_X(k^{r-1}_N)\,, \quad X = k^{r-1}_x N\,,$$

where k^{r-1}_N is represented as a submanifold of $K^{r-1}_n M \subset \widetilde{K}^{r-1}_n M$.
We define $\widetilde{T}^r_n M = \widetilde{J}^r_0(\mathbb{R}^n, M)$. This is extended into a bundle functor $\widetilde{T}^r_n$ in the way of Section 5.4. A natural equivalence of functors

$$\mu^r_M \colon \widetilde{T}^r_n M \to T^1_n(\widetilde{T}^{r-1}_n M)$$

is defined as follows. Every $X \in \widetilde{T}^r_n M$ is of the form $X = j^1_0\varphi$, where $\varphi\colon \mathbb{R}^n \to \widetilde{J}^{r-1}(\mathbb{R}^n, M)$ is a section of the source projection $\widetilde{J}^{r-1}(\mathbb{R}^n, M) \to \mathbb{R}^n$. On the other hand, $Z \in T^1_n(\widetilde{T}^{r-1}_n M)$ means $Z = j^1_0\psi$ with $\psi\colon \mathbb{R}^n \to \widetilde{J}^{r-1}(\mathbb{R}^n, M)$. Write $t_u\colon \mathbb{R}^n \to \mathbb{R}^n$ for the translation $x \mapsto x + u$. Then $u \mapsto (\varphi(u) \circ j^{r-1}_u t^{-1}_u)$ is a map $\mathbb{R}^n \to \widetilde{J}^{r-1}_o(\mathbb{R}^n, M)$ and we set

$$\mu^r_M(X) = j^1_0\big(\varphi(u) \circ j^{r-1}_u t^{-1}_u\big)\,. \tag{8.2}$$

One verifies easily that μ^r_M maps $\operatorname{reg} \widetilde{T}^r_n M$ into $\operatorname{reg} T^1_n(\operatorname{reg} \widetilde{T}^{r-1}_n M)$.

On every fibered manifold $p\colon Y \to M$, a contact element $X \in K^1_n Y$ is said to be transversal, if the underlying linear n-space of X has zero intersection with the vertical tangent space of Y. We write $\operatorname{tr} K^1_n Y \subset K^1_n Y$ for the subset of all transversal contact $(n,1)$-elements on Y. This is an open subset of $K^1_n Y$ and $\operatorname{tr} K^1_n Y \to Y$ is a fibered manifold. For $r \geq 2$, the bundle $\operatorname{tr} \widetilde{K}^r_n M \subset \widetilde{K}^r_n M$ of nonholonomic transversal contact (n,r)-elements on M is defined by the iteration

$$\operatorname{tr} \widetilde{K}^r_n M = \operatorname{tr} K^1_n(\operatorname{tr} \widetilde{K}^{r-1}_n M \to M)\,.$$

We recall that $\widetilde{G}^r_n = \operatorname{reg} \widetilde{J}^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$ is a group with respect to the composition of nonholonomic jets.

Proposition. *We have*

$$\operatorname{tr} \widetilde{K}^r_n M = \operatorname{reg} \widetilde{T}^r_n M / \widetilde{G}^r_n\,. \tag{8.3}$$

We write $k\colon \operatorname{reg} \widetilde{T}^r_n M \to \operatorname{tr} \widetilde{K}^r_n M$ for the factor projection.

Proof. Assume by induction $\operatorname{tr} \widetilde{K}^{r-1}_n M = \operatorname{reg} \widetilde{T}^{r-1}_n M / \widetilde{G}^{r-1}_n$. So we have defined $k\colon \operatorname{reg} \widetilde{T}^{r-1}_n M \to \operatorname{tr} \widetilde{K}^{r-1}_n M$. Consider $X \in \operatorname{reg} T^1_n(\operatorname{reg} \widetilde{T}^{r-1}_n M)$, $X = j^1_0\varphi(u)$, $\varphi\colon \mathbb{R}^n \to \widetilde{T}^{r-1}_n M$. Then $u \mapsto k(\varphi(u))$ is the parametrization of an n-dimensional submanifold of $\operatorname{tr} \widetilde{K}^{r-1}_n M$ that is transversal to $\operatorname{tr} \widetilde{K}^{r-1}_n M \to M$. Hence we have $k(j^1_0(k(\varphi(u)))) \in \operatorname{tr} \widetilde{K}^r_n M$. Consider another $\psi(v)\colon \mathbb{R}^n \to \widetilde{T}^{r-1}_n M$ such that $k(j^1_0 k(\psi(v))) = k(j^1_0 k(\varphi(u)))$. First of all, there is a map $v = f(u)$ such that

$$j^1_0 k\big(\psi(f(u))\big) = j^1_0 k\big(\varphi(u)\big)\,.$$

By the induction hypothesis, there is a map $g\colon \mathbb{R}^n \to \widetilde{G}^{r-1}_n$ such that $\psi(f(u)) \circ g(u) = \varphi(u)$. We have $j^1_0 f \in G^1_n$ and $j^1_0 g \in T^1_n \widetilde{G}^{r-1}_n$ and our construction is in accordance with the expression $\widetilde{G}^r_n = G^1_n \rtimes T^1_n \widetilde{G}^{r-1}_n$. $\square$

8.2 Some identifications

Definition. The bundle of semiholonomic contact (n,r)-elements $\overline{K}^r_n M \subset \widetilde{K}^r_n M$ is the subset of all $k^1_x Q$ such that $Q \subset \overline{K}^{r-1}_n M$,
$X = k^1_{\beta_{r-1}(x)}(\beta_{r-1}(Q))$, where $\beta_{r-1}\colon \overline{K}^{r-1}_n M \to \overline{K}^{r-2}_n M$ is defined in the induction procedure starting with the bundle projection $K^1_n M \to M$.

We easily deduce $\overline{K}^r_n M \subset \operatorname{tr} \overline{K}^r_n M$ and $K^r_n M \subset \overline{K}^r_n M$. Analogously to Proposition 8.1, we obtain

$$\overline{K}^r_n M = \operatorname{reg} \overline{T}^r_n M / \overline{G}^r_n \,, \tag{8.4}$$

where $\overline{G}^r_n = \operatorname{reg} \overline{J}^r_0(\mathbb{R}^n, \mathbb{R}^n)_0$ is a subgroup of $\widetilde{G}^r_n$. The canonical projection to lower order semiholonomic contact elements will be denoted by the same symbol π^r_s, $s < r$, as in the jet case.
The underlying contact $(n,1)$-element $X \in (K^r_n \mathbb{R}^m)_x$ is identified with a linear n-space in $T_x\mathbb{R}^m$. Write $\mathbb{R}^{n,m-n}$ for the product bundle $\mathbb{R}^n \times \mathbb{R}^{m-n}$ and $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$ for its bundle projection. Denote by $\tau K^r_n \mathbb{R}^m \subset K^r_n \mathbb{R}^m$ the open subset of all X such that X_1 is transversal to π. It is well known that $\tau K^r_n \mathbb{R}^m$ is identified with the jet prolongation $J^r \mathbb{R}^{n,m-n}$, [KMS].
In the nonholonomic case $X \in \widetilde{K}^r_n \mathbb{R}^m$, we have r underlying contact $(n,1)$-elements $X^{(1)}_1, \dots, X^{(r)}_1$. Write $\tau \widetilde{K}^r_n \mathbb{R}^m \subset \widetilde{K}^r_n \mathbb{R}^m$ for the open subset of all X such that all $X^{(1)}_1, \dots, X^{(r)}_1$ are transversal to π.

Proposition. *$\tau \widetilde{K}^r_n \mathbb{R}^m$ is identified with the r-th nonholonomic prolongation $\widetilde{J}^r \mathbb{R}^{n,m-n}$.*

Proof. By Proposition 8.1, $X \in \tau \widetilde{K}^r_n \mathbb{R}^m$ can be expressed as $X = Z \circ \widetilde{G}^r_n$ with $Z \in (\operatorname{reg} \widetilde{T}^r_n \mathbb{R}^m)_x$. Write $Z^{(1)}_1, \dots, Z^{(r)}_1$ for the underlying 1-velocities of Z. Then $j^1_x \pi \circ Z^{(1)}_1, \dots, j^1_x \pi \circ Z^{(r)}_1$ are invertible 1-jets, so that $\zeta := (j^r_x \pi) \circ Z \in \widetilde{J}^r_0(\mathbb{R}^n, \mathbb{R}^n)$ is invertible. Hence $Z \circ \zeta^{-1}$ satisfies $(j^r_0 \pi) \circ (Z \circ \zeta^{-1}) = j^r_{\pi(x)} \operatorname{id}_{\mathbb{R}^n}$, which implies $Z \circ \zeta^{-1} \in \widetilde{J}^r \mathbb{R}^{n,m-n}$. □

In the semiholonomic case $X \in \overline{K}^r_n \mathbb{R}^m$, we have $X^{(1)}_1 = \cdots = X^{(r)}_1 =: X_1$. We write $\tau \overline{K}^r_n \mathbb{R}^m \subset \overline{K}^r_n \mathbb{R}^m$ for the open subset of all X such that X_1 is transversal to $\pi\colon \mathbb{R}^m \to \mathbb{R}^n$. In the same way as in the above proposition, we construct an identification

$$\tau \overline{K}^r_n \mathbb{R}^m \approx \overline{J}^r \mathbb{R}^{n,m-n} \tag{8.5}$$

where $\overline{J}^r \mathbb{R}^{n,m-n}$ denotes the r-th semiholonomic prolongation of $\mathbb{R}^{n,m-n}$. In particular, for $n = 1$ we have $\overline{K}^r_1 M = K^r_1 M$.

Remark. Some further properties of the iterated contact elements are deduced by using the general concept of contact (n, F)-element for a regular subcategory F of the category of nonholonomic r-jets in [Ko11]. Special attention is paid to the incidence relation of contact F-elements of different dimensions.

8.3 The absolute contact differentiation

Consider a principal bundle $P(M, G)$ with a principal connection Γ, a left G-space S and the associated bundle $E = P[S]$. For every section s of E, its absolute differential can be viewed as a section

$$\nabla_\Gamma s \colon M \to J^1_x(M, E_x)_{s(x)} \,.$$

If $\Gamma(u) = j^1_x \varrho$ for a local section ϱ of P, then $(\nabla_\Gamma s)$ is transformed by $\widetilde{u}^{-1}$ into

$$j^1_x\big(\widetilde{\varrho(y)}^{-1}(s(y))\big) \in J^1_x(M, S)\,, \quad y \in M\,, \tag{8.6}$$

where $\widetilde{u} \colon S \to E_x$ denotes the frame map corresponding to $u \in P$.
First we introduce the general concept of absolute contact differential (a concrete version of which will be discussed in more details in Section 9.2). Replace M by N and assume $n = \dim N < \dim S$. Clearly, all spaces $K^1_n(E_x)$, $x \in N$, form an associated bundle

$$\bigcup_{x \in N} K^1_n(E_x) = P\big[K^1_n S\big] \,.$$

Assume further, that each $\nabla_\Gamma s(x)$ is a regular 1-jet.

Definition. The absolute contact differential $k\nabla_\Gamma$ of s is defined by

$$\big((k\nabla_\Gamma)s\big)(x) = k\big((\nabla_\Gamma s)(x)\big)\,, \quad x \in N\,. \tag{8.7}$$

Hence $k\nabla_\Gamma s$ is a section $N \to P[K^1_n S]$. Since we have a section of another bundle associated to P, we can construct

$$\nabla_\Gamma\big((k\nabla^2_\Gamma)s\big)(x) = k\big(\nabla_\Gamma((k\nabla_\Gamma)s)(x)\big)\,. \tag{8.8}$$

This is formed by regular 1-jets and we define

$$\big((k\nabla^2_\Gamma)s\big)(x) = k\big(\nabla_\Gamma((k\nabla_\Gamma)s)\big)(x)\,. \tag{8.9}$$

One verifies easily that this is an element of $\bar{K}^2_n(E_x)$. Hence $(k\nabla^2_\Gamma)s$ is a section $N \to P[\bar{K}^2_n S]$.

By iteration, the r-th absolute contact differential of s is defined as

$$\big((k\nabla^r_\Gamma)s\big)(x) = k\big(\nabla_\Gamma((k\nabla^{r-1}_\Gamma)s)(x)\big)\,. \tag{8.10}$$

By the very definition of semiholonomic contact (n,r)-elements, we deduce that (8.10) form a section

$$(k\nabla^r_\Gamma s)\colon N \to P[\overline{K}^r_n S]\,.$$

Remark. If Γ is curvature free, then P can be locally viewed as the product $N \times G$ with the canonical flat connection. Then (8.10) implies that the values of $(k\nabla^r_\Gamma)s$ are holonomic contact elements for every section $s\colon M \to E$.

8.4 The first jet prolongation of sections of associated bundles

According to [KMS], it is useful to start with the case of the first jet prolongation of a section $s\colon M \to E$. Let $P(M,G)$ be a principal fiber bundle and l be a left action of G on a manifold S. The associated bundle $E = P[S]$ is defined by the equivalence relation

$$(u,y) \sim \big(ug, l(g^{-1})(y)\big)\,, \quad u \in P, g \in G, y \in S\,.$$

Hence every $u \in P_x$ can be interpreted as a frame map $\widetilde{u}\colon S \to E_x$, $\widetilde{u}(y) = \{u,y\}$.

It is well known that the canonical $\mathbb{R}^m$-valued one-form $\varphi\colon TP^1M \to \mathbb{R}^m$ is defined as follows. Every $u \in P^1_xM$ is a linear isomorphism $\overline{u}\colon \mathbb{R}^m \to T_xM$ and for every $X \in T_uP^1M$ we set

$$\varphi(X) := \overline{u}^{-1}\big(T\beta(X)\big)\,, \quad \beta\colon P^1M \to M\,. \tag{8.11}$$

In the case of W^1P, we define analogously a canonical $(\mathbb{R}^m \times \mathfrak{g})$-valued form θ on W^1P. Every $u \in W^1_xP$ is a linear isomorphism $\overline{u}\colon \mathbb{R}^m \times \mathfrak{g} \to T_{\beta(u)}P$ and for every $X \in T_uW^1P$ we set

$$\theta(X) = \overline{u}^{-1}\big(T\beta(X)\big)\,, \quad \beta\colon W^1P \to P\,. \tag{8.12}$$

Let e_i, $i = 1,\dots,m$ be the canonical basis of $\mathbb{R}^m$, and e_α, $\alpha = m+1,\dots,m+\dim G$ be a linear basis of $\mathfrak{g}$. Consider further the fundamental vector fields on S determined by e_α. Hence the coordinate form of Y_α is $Y_\alpha = \eta^p_\alpha(y^q)\frac{\partial}{\partial y^p}$.
The following assertion can be found in [KMS], p. 157.

Proposition. *Let a^p be the coordinate functions of a section $s\colon M \to E$ and a^p_i be the additional coordinate functions of the first jet prolongation $j^1 s$. Then*

$$da^p + \eta^p_\alpha(a^q)\theta^\alpha = a^p_i \theta^i \,, \tag{8.13}$$

where θ^i and θ^α are the coordinate components of θ.

8.5 The first order absolute differentiation

We are going to deduce a general coordinate formula for the absolute differentiation on an arbitrary associated bundle $E = P[S]$. Consider a principal connection $\Gamma\colon P \to J^1P$ on a principal bundle $P(M,G)$. Clearly, Γ defines a reduction

$$R(\Gamma) = P^1M \times_M P \to P^1M \times_M \Gamma(P) \hookrightarrow P^1M \times_M J^1P \tag{8.14}$$

of principal bundle W^1P to the structure group $G^1_m \times G \subset W^1_m G$ with a direct product of G^1_m and G.
Let us write $\widetilde{\theta}$ for the restriction of the canonical form θ of W^1P to $R(\Gamma)$. We have $\widetilde{\theta} = \theta_M \times \omega_\Gamma$, where $\theta_M\colon TP^1M \to \mathbb{R}^m$ is the canonical form of P^1M and $\omega_\Gamma\colon TP \to \mathfrak{g}$ is the well known connection form of Γ.
Consider an associated bundle $E = P[S]$ as in Section 8.4. Then $\nabla_\Gamma s$ is identified with a section of $P[T^1_m S]$. Clearly, we may consider $\nabla_\Gamma s$ as a section of the associated bundle $R(\Gamma)[T^1_m S]$ in the sense of Section 6.5. Write ω^α for the coordinate components of ω.

Proposition. *Let a^p be the coordinate functions of a section $s\colon M \to E$ and a^p_i be the additional coordinate functions of $\nabla_\Gamma s$ on $P^1M \times_M P$. Then*

$$da^p + \eta^p_\alpha(a^q)\omega^\alpha = a^p_i \theta^i_M \,. \tag{8.15}$$

Proof. We restrict the formula (8.13) from W^1P to $R(\Gamma)$. □

8.6 The first order absolute contact differentiation

Since $K^1_n S$, $n < m$, is the manifold of all linear n-spaces in the tangent spaces to S, this is also a G-space. By Section 8.5, it suffices to determine the induced action $K^1_n l\colon G \times K^1_n S \to K^1_n S$, that is projectable over $l\colon G \times S \to S$. This can be performed by the standard technique of linear algebra. But we do not need an explicit formula for $K^1_n l$. The geometrically most interesting case of the submanifolds of Cartan spaces will be discussed in the next chapter in a separate way.

Chapter 9

Submanifolds of Cartan spaces

9.1 Cartan spaces

We recall that a Klein space is a manifold S with a transitive left action $(g, x) \mapsto gx$ of a Lie group G. Fix a point $c \in S$ and write H for its stability group. Then S coincides with the coset space $S = G/H$, $c = \{H\}$ and G can be viewed as a principal H-bundle over S with bundle projection $g \mapsto gc$. Every $g \in G(S, H)$ is interpreted as a frame $\widetilde{g}\colon S \to S$, $\widetilde{g}(a) = ga$, $a \in S$.
A "curved" version of S can be defined in two formally different ways. First we present the viewpoint from the book by Sharpe, [Sh]. Consider a pair (G, H) of a Lie group G and a closed subgroup $H \subset G$.

Definition (a). Cartan geometry of type (G, H) is a principal bundle $Q(M, H)$ with 1-form $\omega\colon TQ \to \mathfrak{g}$ (which is said to be Cartan connection) such that
(i) $\omega(u)\colon T_uQ \to \mathfrak{g}$ is a linear isomorphism for every $u \in Q$,
(ii) $(R_h)^*\omega = Ad(h^{-1}) \circ \omega$ for every $h \in H$,
(iii) $\omega(X^*(u)) = X$ for every $X \in \mathfrak{h}$ and every $u \in Q$, where X^* is the fundamental vector field on Q induced by X.
We remark that, in addition to [Sh], further interesting examples of Cartan spaces can be found in [CaSl].
In what follows, we assume G acts effectively on the coset space G/H. So $S = G/H$ is a Klein space. Clearly, $T_cS = \mathfrak{g}/\mathfrak{h}$.
On the other hand, consider $P(M, G)$, S, $E = P[S]$ as in Section 8.3 and fix a section $s\colon M \to E$. The following definition in [KoVi] was based directly

on some ideas by Ehresmann, [Eh].

Definition (b). Consider a Klein space $S = G/H$. A space with Cartan connection of type (G, H) over M is a quadruple $S = \mathcal{S}(M) = (P(M,G), \Gamma, E = P[S], s)$ such that $\dim M = \dim S$ and the absolute differential $\nabla_\Gamma s$ is formed by regular 1-jets.

We deduce that both concepts are naturally equivalent. In the case (b), s defines a reduction to subgroup H

$$Q = \{u \in P, \widetilde{u}(c) = s(p(u))\},$$

where $p\colon P \to M$ is the bundle projection. Write ω for the restriction of the connection form $\omega_\Gamma\colon TP \to \mathfrak{g}$ to Q. Clearly, $\omega\colon TQ \to \mathfrak{g}$ satisfies (ii) and (iii) from Definition (a).

Lemma. *ω satisfies (i), iff the 1-jets $(\nabla_\Gamma s)(x)$ are regular for all $x \in M$.*

Proof. The vertical tangent bundle VE is an associated bundle $P[TS]$ and $T\widetilde{u}\colon TS \to TE_x$ is the induced frame map on VE. Consider $X = \frac{d\gamma(0)}{dt} \in T_xM$, $\gamma\colon \mathbb{R} \to M$. Then

$$(T\widetilde{y})^{-1}\big((\nabla_\Gamma s)(X)\big) = \frac{d}{dt}\Big|_0 \widetilde{\varrho(\gamma(t))}^{-1}\big(s(\gamma(t))\big).$$

Since G acts transitively on S, we have

$$\widetilde{\varrho(\gamma(t))}^{-1}\big(s(\gamma(t))\big) = \delta(t)c, \quad \delta\colon \mathbb{R} \to G.$$

Write $Z = \frac{d}{dt}\big|_0 \varrho\big(\gamma(t)\big)\delta(t) \in T_uQ$. By the definition of the connection form, we have

$$(T\widetilde{u})^{-1}\big((\nabla_\Gamma s)(X)\big) = \omega(Z) + \mathfrak{h} \in T_cS.$$

These vectors are linearly independent for a basis of T_xM, iff $\omega(u)$ is a linear isomorphism. □

Using the above lemma one easily verifies that Definitions (a) and (b) are equivalent. We shall say that $\mathcal{S}(M)$ is a Cartan space and ω will be called its connection form. Clearly, the connection form of a Klein space $S = G/H$ is the Maurer-Cartan form $\omega\colon TG \to \mathfrak{g}$.

9.2 The absolute contact differentiation on submanifolds

Consider an n-submanifold $N \subset M$. If we restrict all objects in question over N, we obtain

$$(P_N, \Gamma_N, E_N, s_N) = \mathcal{S}(N) = (Q_N, \omega_N)\,.$$

Then we have the situation from Section 8.3. By induction we deduce that $(k\nabla^r_{\Gamma_N})s_N$ depends on $k^r_x N$ only . Write $S^r_n = (K^r_n S)_c$ and $\overline{S}^r_n = (\overline{K}^r_n S)_c$. Clearly, both S^r_n and $\overline{S}^r_n$ are H-spaces.
Further, the map

$$\Gamma^r_n \colon K^r_n M \to Q[\overline{S}^r_n]\,, \quad \Gamma^r_n(k^r_x N) = (k\nabla^r_{\Gamma_N})s_N(x) \tag{9.1}$$

is called the formal absolute contact (n, r)-differentiation on $\mathcal{S}(M)$. The map

$$\Gamma^r_N = \Gamma^r_n \circ k^r_N \colon N \to Q_N[\overline{S}^r_n] \tag{9.2}$$

is said to be the r-th absolute contact differential of N.
If we discuss the Klein space G/H with the canonical flat connection, we have $Q[S^r_n] = K^r_n S$. Then, by Remark 8.3, Γ^r_n is the identity of $K^r_n S$ composed with the injection $K^r_n S \hookrightarrow \overline{K}^r_n S = Q[\overline{S}^r_n]$.

9.3 Submanifolds of Klein spaces

In general, if we have a section $s \colon M \to P[S]$ of an associated bundle, the map $P \to S$,

$$u \mapsto \widetilde{u}\big(s(p(u))\big) \tag{9.3}$$

is said to be the frame form of s. Given some local coordinates y^p on S, the locally defined compositions of (9.3) with y^p are called the coordinate functions of s.
If $N \subset M$ is an n-dimensional submanifold of a Klein space $S = G/H$ over M, we write G_N for the restriction of principal bundle $G(S, H)$ over N. Then k^r_N is a section $N \to G_N[S^r_n]$, that is sometimes called the fundamental r-th order field of N. It was pointed out by G.F Laptěv, [La] (but in the coordinate form only), that a modification of the Cartan method of moving frames leads to the coordinate functions of k^r_N.
The elements of G_N are said to be the zero order frames of N. They are characterized by the property that the images $\widetilde{g}(c)$ of $c \in S$ under the frame map $\widetilde{g}$ lie in N. So the frame form of k^r_N is a map $G_N \to S^r_n$.

Consider the canonical coordinates x^i, y^p on $\mathbb{R}^{n,m-n}$. The induced coordinates on the r-th jet prolongation $J^r\mathbb{R}^{n,m-n}$ are

$$y^p_i, y^p_{ij}, \dots y^p_{i_1\dots i_r}\,. \tag{9.4}$$

Choose a local coordinate system x^i, y^p on S centered at c. This identifies locally S^r_n with $J^r_0\mathbb{R}^{n,m-n}$. We write

$$(a^p_i, a^p_{ij}, \dots, a^p_{i_1\dots i_r})\colon G_N \to S^r_n \tag{9.5}$$

for the locally defined coordinate functions of the section k^r_N of $G_N[S^r_n]$.
The algorithm for finding (9.5) by a Cartan-like procedure from [La] is described in [Ko77]. This general approach is based on the use of zero order frames of N. However, the evaluations in zero order frames are top-heavy because of the nontrivial topological character of the classical Grassmann manifolds. Thus, in practice one always uses the first order frames of N. So, also here we will consider the first order frames.
Assume further that H acts transitively on S^1_n, which is satisfied for all classical Klein spaces. Choose a point $c_n \in S^1_n$, write H_1 for its stability group and S^r_{n1} for the fiber of $S^r_n \to S^1_n$ over c_n. Clearly, S^r_{n1} is an H-space. A frame $\widetilde{g} \in G_N$ is said to be the first order frame of N, if $\widetilde{g}(c_n) = T_{gc}N$. Clearly, the space G_{N1} of all first order frames on N is a principal bundle $G_{N1}(N, H_1)$. It we restrict ourselves to the first order frames, the frame form of k^r_N is a map $G_{N1} \to S^r_{n1}$. Assume further that the equations of c_n are $dy^p = 0$. In other words, in the jet coordinates of k^r_N we have $y^p_i = 0$. So the first order frames of N are characterized by $a^p_i = 0$. The induced global coordinates on S^r_{n1} are $y^p_{ij}, \dots, y^p_{i_1\dots i_r}$. If we interpret k^r_N as a section of the associated bundle $G_{N1}[S^r_{N1}]$, then its coordinate functions

$$(a^p_{ij}, \dots, a^p_{i_1\dots i_r})\colon G_{N1} \to S^r_{n1} \tag{9.6}$$

are globally defined.
The simpliest algorithm appears in the case there exists an Abelian subgroup $K \subset G$ such that $\mathfrak{g}$ is the product $\mathfrak{k} \times \mathfrak{h}$. But all classical Klein spaces have this property. (For example, if A_m is an m-dimensional affine space, we have $G = GA(m)$, $H = GL(m)$ and $K = \mathbb{R}^m \subset GA(m)$ is the Abelian subgroup of all translations on A_m.) We choose a basis of $\mathfrak{g}$

$$e_\alpha, e_\lambda \quad \alpha, \beta = 1, \dots, m, \quad \lambda, \mu, \nu = m+1, \dots, \dim G \tag{9.7}$$

such that e_λ lie in $\mathfrak{h}$ and e_α is a basis of $\mathfrak{k}$.

This assumption is equivalent to the following relations on the structure constants of G

$$c^{\alpha}_{\beta\gamma} = 0\,, \quad c^{\lambda}_{\alpha\beta} = 0\,, \quad c^{\alpha}_{\lambda\mu} = 0\,. \tag{9.8}$$

Hence the coordinate form of the structure equation $d\varphi + \frac{1}{2}[\varphi, \varphi] = 0$ of the Maurer-Cartan form φ is

$$\begin{aligned} d\varphi^{\alpha} &= c^{\alpha}_{\lambda\beta}\varphi^{\beta} \wedge \varphi^{\lambda}\,, \\ d\varphi^{\lambda} &= c^{\lambda}_{\alpha\mu}\varphi^{\mu} \wedge \varphi^{\alpha} - \frac{1}{2}c^{\lambda}_{\mu\nu}\varphi^{\mu} \wedge \varphi^{\nu}\,. \end{aligned} \tag{9.9}$$

We shall write π^{λ} for the restriction of φ^{λ} to H. The bundle projection $G \to S$ identifies locally K with S. So the basis e_{α} defines local coordinates x^{α} on S with $(x^{\alpha}) = (x^i, y^p)$.
In what follows we shall write φ for the restriction φ_{N1} of φ to G_{N1}, as usual in concrete investigations. So our starting point are the equations

$$\varphi^p = 0\,. \tag{9.10}$$

If we substitute them into (9.9), we obtain

$$0 = c^p_{i\lambda}\varphi^{\lambda} \wedge \varphi^i\,. \tag{9.11}$$

Using the Cartan lemma, we find

$$c^p_{i\lambda}\varphi^{\lambda} = a^p_{ij}\varphi^j \qquad a^p_{ij} = a^p_{ji}\,. \tag{9.12}$$

In [Ko77], we deduced that a^p_{ij} coincide with the coordinate functions of k^2_N on G_{N1}.
In particular, (9.12) implies that the differential equations of H_1 are

$$c^p_{i\lambda}\pi^{\lambda} = 0\,.$$

Now we apply the exterior differentiation to (9.12). Using the structure equations, we obtain an expression of the form

$$[da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl})\varphi^{\lambda}] \wedge \varphi^j = 0\,. \tag{9.13}$$

If we apply the Cartan lemma to (9.13), we obtain

$$da^p_{ij} - \Phi^p_{ij\lambda}(a^q_{kl})\varphi^{\lambda} = a^p_{ijk}\varphi^k\,. \tag{9.14}$$

(We shall see that a^p_{ijk} are the addition coordinate functions of k^3_N on G_{N1}.)

This procedure can be iterated. Assume by induction that after $r-3$ steps we have the equations of the infinitesimal action of H_1 on S^{r-2}_{n1}

$$\begin{aligned} dy^p_{ij} - \Phi^p_{ij\lambda}(y^q_{kl})\pi^\lambda &= 0 \\ &\vdots \\ dy^p_{i_1\dots i_{r-2}} - \Phi^p_{i_1\dots i_{r-2}\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-2}})\pi^\lambda &= 0 \end{aligned} \tag{9.15}$$

with $c^p_{i\lambda}\pi^\lambda = 0$, and it holds

$$\begin{aligned} c^p_{i\lambda}\varphi^\lambda &= a^p_{ij}\varphi^j\,, \\ &\vdots \\ da^p_{i_1\dots i_{r-2}} - \Phi^p_{i_1\dots i_{r-2}\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-2}})\varphi^\lambda &= a_{i_1\dots i_{r-2}j}\varphi^j\,. \end{aligned} \tag{9.16}$$

If we apply exterior differentiation to the last row and use all these equations, we obtain certain relations of the form

$$\left[da^p_{i_1\dots i_{r-2}k} - \Phi^p_{i_1\dots i_{r-2}k\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-1}})\varphi^\lambda\right]\wedge\varphi^k = 0 \tag{9.17}$$

In [Ko77] we deduced

Proposition. *The additional equations of the infinitesimal action of H_1 on S^{r-1}_{n1} are*

$$dy^p_{i_1\dots i_{r-1}} - \Phi^p_{i_1\dots i_{r-1}\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-1}})\pi^\lambda = 0 \quad \textit{with} \quad c^p_{i\lambda}\pi^\lambda = 0\,. \tag{9.18}$$

The additional coordinate functions $a^p_{i_1\dots i_r}$ of k^r_N on G_{N1} satisfy

$$da^p_{i_1\dots i_{r-1}} - \Phi_{i_1\dots i_{r-1}\lambda}(a^q_{j_1j_2},\dots,a^q_{j_1\dots j_{r-1}})\varphi^\lambda = a^p_{i_1\dots i_r}\varphi^{i_r}\,. \tag{9.19}$$

□

The Cartan method of moving frames is usually used for finding differential invariants of $N\subset S$ and for solving the equivalence problem for N. The fact that the above procedure yields the equations of the infinitesimal action of H_1 on S^r_{n1} was used in [La] for local computation of the geometric objects of N. Our analysis of these algorithms led us to the following conceptual definition, [Ko77]. Let Z be an H-space.

Definition. A geometric (n,r)-object on S is an H-equivariant map $\mu\colon S^r_n\to Z$.

Since μ is an H-map, it induces the associated bundle morphism $\widetilde{\mu}\colon K_n^r S \to G[Z]$. The map

$$\mu_N = \widetilde{\mu} \circ k_N^r \colon N \to G_N[Z] \tag{9.20}$$

is called the value of geometric (n,r)-object μ on N. More generally, let $W \subset S_n^r$ be an H-invariant submanifold. An n-submanifold $N \subset S$ is said to be of type W, if the values of k_N^r lie in $G_N[W]$. We can introduce a geometric object of type W as an H-map $W \to Z$. For a submanifold N of type W, $\mu \circ k_N^r$ is the value of μ on N. Very simple examples of W are the elliptic, parabolic and hyperbolic contact $(2,2)$-elements on Euclidean 3-space.

The equations of the infinitesimal action can be used, at least locally, for constructing the equivariant maps. A general global result is due to R. Palais, [Pa].

In practice, one constructs the geometric objects of N by using the first order frames. If we interpret H as a principal H_1-bundle $H(H/H_1, H_1)$, then S_n^r coincides with the associated bundle $H[S_{n1}^r]$. The left action of H on S_n^r has the form

$$h_1[h_2, y] = [h_1 h_2, y]\,, \quad h_1, h_2 \in H, y \in S_{n1}^r\,.$$

Let B be an H-space. The associated bundle $H[B]$ is an H-space with respect to the action

$$h_1[h_2, z] = [h_1 h_2, z]\,, \quad h_1, h_2 \in H, z \in B\,.$$

This definition is correct, for $h_3\{h_1 h_2, h_2^{-1} z\} = \{h_3 h_1 h_2, h_2^{-1} z\} = h_3\{h_1, z\}$, $h_1 \in H$. For every H_1-map $\nu\colon S_{n1}^r \to B$, the induced map $\widetilde{\nu}\colon H[S_{n1}^r] \to H[B]$ is H-equivariant. So every H_1-map $\nu\colon S_{n1}^r \to B$ gives rise to a geometric (n,r)-object on S.

We underline that the concept of r-th order geometric object for n-submanifolds of S is of universal character. Its specification to an n-submanifold $N \subset S$ (or to a submanifold of type W) is constructed by means of the contact elements, so that it is independent of parametrizations of N.

The differential invariants of submanifolds are the simpliest example of geometric objects. In this case, $Z = \mathbb{R}$ with the identity action of H. Further, if we consider the action of H on $\mathbb{R}$ by means of homotheties, we obtain the so-called relative invariants.

Remark. According to the Cartan-like algorithm of this section, the geometric objects of a submanifold $N \subset S$ are determined by the restriction

φ_N of the Maurer-Cartan form of G over N. This corresponds to the well-known role of φ_N in the equivalence problem for N, [Ca37a]. We recall that this role is based on the fact that, for a connected manifold N, two maps f_1, $f_2\colon N \to G$ are congruent, i.e. there exists $g \in G$ such that$f_1(x) = gf_2(x)$ for all $x \in N$, if and only if $\varphi \circ Tf_1 = \varphi \circ Tf_2 \colon TN \to \mathfrak{g}$.

9.4 Submanifolds of Cartan spaces

Consider a Cartan space $\mathcal{S}(M)$ such that H acts transitively on S^1_n. Let $N \subset M$ be an n-submanifold. The elements of Q_N are zero order frames of N, they are characterized by $\widetilde{u}(c) \in s_N(N)$. A frame $u \in Q_N$ is a first order frame of N, if $\omega^p_N(u) = 0$. Analogously to Section 9.3, there frames form a reduction Q_{N1} of Q_N to H_1. We write $\overline{S}^r_n = (\overline{K}^r_n S)_c$. Then $\overline{K}^r_n S$ is an associated bundle $G[\overline{S}^r_n]$. Further, we write $\overline{S}^r_{n1}$ for the fiber of $\overline{S}^r_n \to S^1_n$ over c_n.
The r-th absolute contact differential Γ^r_N can be viewed as a section of the associated bundle $Q_{N1}[\overline{S}^r_{n1}]$. By Section 8.2, the coordinates x^i, y^p identify $\overline{S}^r_n$ locally with $\overline{J}^r_0\mathbb{R}^{n,m-n}$. Hence the induced coordinates on $\overline{S}^r_{n1}$ are

$$y^p_{ij}, \dots, y^p_{i_1\dots i_r} \tag{9.21}$$

arbitrary in all subscripts. The coordinate functions of Γ^r_N

$$(b^p_{ij}, \dots, b^p_{i_1\dots i_r})\colon Q_{N1} \to \overline{S}^r_{n1} \tag{9.22}$$

are globally defined.
Assume the existence of $K \subset G$ as in Section 9.3. Then we have the following simple procedure for finding the coordinate functions of Γ^r_N, in which the role of the Maurer-Cartan form φ from Section 9.3 is replace by the connection form ω. We write ω for the restriction ω_{N1} of ω to Q_{N1}. So our starting point are the equations

$$\omega^p = 0\,.$$

In [Ko73], we deduced
Proposition (a). *We have*

$$c^p_{i\lambda}\omega^\lambda = b^p_{ij}\omega^j\,. \tag{9.23}$$

□

Now we proceed by induction. Assume that after $r-3$ steps we have deduced the equations of the infinitesimal action of H_1 on $\overline{S}^{r-2}_{n1}$

$$\begin{aligned} &dy^p_{ij} - \Psi^p_{ij\lambda}(y^q_{kl})\pi^\lambda = 0 \\ &\vdots \\ &dy^p_{i_1\dots i_{r-2}} - \Psi^p_{i_1\dots i_{r-2}\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-2}})\pi^\lambda = 0 \end{aligned} \tag{9.24}$$

with $c^p_{i\lambda}\pi^\lambda = 0$. Write formally the relations

$$\begin{aligned} &dy^p_{ij} - \Psi^p_{ij\lambda}(y^q_{kl})\varphi^\lambda = y^p_{ijk}\varphi^k\,, \\ &\vdots \\ &dy^p_{i_1\dots i_{r-2}} - \Psi_{i_1\dots i_{r-2}\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-2}})\varphi^\lambda = y_{i_1\dots i_{r-1}i_{r-1}}\varphi^{i_{r-1}} \end{aligned} \tag{9.25}$$

as well as

$$dy^p_{i_1\dots i_{r-2}j} - \Psi^p_{i_1\dots i_{r-2}j\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-1}})\varphi^\lambda = y^p_{i_1\dots i_{r-2}j}\varphi^j\,. \tag{9.26}$$

Applying exterior differentiation to (9.25) and (9.26) with $\varphi^p = 0$, using the structure equations of φ and (9.26), we obtain an expression of the form

$$\left[dy^p_{i_1\dots i_{r-2}j} - \Psi^p_{i_1\dots i_{r-2}j\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-1}})\varphi^\lambda\right]\wedge\varphi^j = 0\,. \tag{9.27}$$

Proposition (b). *The additional equations of the infinitesimal action of H_1 on $\overline{S}^r_{n1}$ are*

$$dy^p_{i_1\dots i_{r-1}j} - \Psi_{i_1\dots i_{r-1}\lambda}(y^q_{j_1j_2},\dots,y^q_{j_1\dots j_{r-1}})\pi^\lambda = 0 \quad \textit{with} \quad c^p_{i\lambda}\pi^\lambda = 0\,. \tag{9.28}$$

The coordinate functions $b^p_{ij},\dots,b^p_{i_1\dots i_r}$ of Γ^r_N on Q_{N1} satisfy (9.23) and

$$\begin{aligned} db^p_{ij} - &\Psi^p_{ij\lambda}(b^q_{kl})\omega^\lambda = b^p_{ijk}\omega^k\,, \\ &\vdots \\ db^p_{i_1\dots i_{r-1}} - &\Psi^p_{i_1\dots i_{r-1}\lambda}(b^q_{j_1j_2},\dots,b^q_{j_1\dots j_{r-1}})\omega^\lambda = b^p_{i_1\dots i_{r-1}j}\omega^j\,. \end{aligned} \tag{9.29}$$

□

Remark. We underline that the absolute contact differential of any order of N is determined by the restriction ω_N of the connection ω over N. This is an important analogy of Remark 9.3. Clearly, Section 9.3 can be viewed as a special case, provided we consider G/H as a flat Cartan space.

Now we generalize the concept of geometric (n,r)-object to Cartan spaces. Let Z be an H-space.

Definition. A geometric (n,r)-object on $\mathcal{S}(M)$ is an H-equivariant map $\mu\colon \overline{S}^r_n \to Z$.

We also say that μ is a semiholonomic (n,r)-object. For $n=1$ we have $\overline{S}^r_1 = S^r_1$, so that there exist holonomic $(1,r)$-objects only.
So we have the induced bundle morphism $\mu\colon Q[\overline{S}^r_n] \to Q[Z]$. For a submanifold $N \subset M$, the composition

$$\mu_N\colon \overline{\mu} \circ \Gamma^r_N\colon N \to Q_N[Z]$$

is called the value of μ on N. More generally, if $W \subset \overline{S}^r_n$ is an H-invariant submanifold, then the (n,r)-objects of type W are defined analogously to Section 9.3.

9.5 The torsion-free case

For a Cartan space $\mathcal{S}(M)$, Sharpe defines its curvature Ω by

$$d\omega + \frac{1}{2}[\omega,\omega] = \Omega\,, \tag{9.30}$$

[Sh]. So Ω is the restriction of the curvature Ω_Γ of Γ to Q. It is well known that Ω_Γ can be interpreted as a map

$$\Omega_\Gamma\colon P \times_M \Lambda^2 TM \to \mathfrak{g}\,.$$

Hence we may consider Ω as a map

$$\Omega\colon Q \times_M \Lambda^2 TM \to \mathfrak{g}\,. \tag{9.31}$$

The coordinate form of (9.30) is

$$d\omega^I + \frac{1}{2}c^I_{JK}\omega^J \wedge \omega^K = R^I_{\alpha\beta}\omega^\alpha \wedge \omega^\beta\,, \quad I,J,K = 1,\dots,\dim G\,. \tag{9.32}$$

In [Ko71a] we introduced the following concept in a slighty different, but equivalent way. Write $L = \mathfrak{g}/\mathfrak{h} = T_cS$ and $\nu\colon \mathfrak{g} \to L$ for the factor projection.

Definition. The composition $\sigma = \nu \circ \Omega\colon Q \times_M \Lambda^2 TM \to L$ is called the torsion of $\mathcal{S}(M)$.

The absolute differentiation with respect to Γ identifies T_xM with $T_{s(x)}E_x$. Clearly, L is an H-space and the corresponding associated bundle satisfies

$$Q[L] = \bigcup_{x\in M} T_{s(x)}E_x\,. \tag{9.33}$$

Hence σ can be interpreted as a section

$$\sigma\colon M \to Q[L\otimes\Lambda^2 L^*]\,. \tag{9.34}$$

By (9.32), the coordinate expression of σ is

$$R^\gamma_{\alpha\beta}\omega^\alpha\wedge\omega^\beta\,. \tag{9.35}$$

This implies that σ coincides with the standard torsion in the classical case of a principal connection on the first order frame bundle of M.

Remark. The concept of higher order torsion of Cartan spaces is discussed from a similar point of view in [Ko71a].

Our result from [Ko71a] can be now formulated as follows. (Another approach to this assertion will be discussed in the next section.)

Proposition. *The torsion of $\mathcal{S}(M)$ vanishes, iff the values of Γ^2_n are holonomic contact $(n,2)$-elements.* □

In particular, this is true in the case of a Riemannian manifold (M,g), that is considered as a Cartan space $\mathcal{E}(M)$ with respect to the Levi-Civita connection. Thus, from the viewpoint of our approach, the second-order geometric objects on submanifolds of Riemannian spaces are of the same type as in the case of submanifolds of Euclidean spaces.

9.6 The difference tensor for contact elements

We modify the idea of the difference tensor of semiholonomic 2-jets to semiholonomic contact 2-elements.
Consider $\xi\in\overline{K}^2_nM$ and $X\in\operatorname{reg}\overline{T}^2_nM$ satisfying $\xi=k^2(X)$. The underlying contact 1-element $\xi_1\in K^1_nM$ is identified with a linear subspace $\lambda(\xi_1)\subset T_xM$. Since X is regular, the value of the linear projection $\delta(\xi)$ of $\Delta(X)$ into $T_xM\otimes\Lambda^2\mathbb{R}^{n*}$ lies in $\lambda(\xi_1)\otimes\Lambda^2\mathbb{R}^{n*}$.

Proposition. *$\xi\in\overline{K}^2_nM$ is a holonomic contact 2-element, if and only if $\delta(\xi)=0$.*

Proof. For $r = 2$, the identification (8.4) is of the form

$$\zeta\colon \overline{K}^2_n M \to \operatorname{reg} \overline{T}^2_n M/\overline{G}^2_n\,. \tag{9.36}$$

The relation $G^2_n \subset \overline{G}^2_n$ induces a projection

$$\chi\colon \operatorname{reg} \overline{T}^2_n M/\overline{G}^2_n \to \operatorname{reg} \overline{T}^2_n/G^2_n\,. \tag{9.37}$$

Hence we can construct

$$\chi \circ \zeta\colon \overline{K}^2_n M \to \operatorname{reg} \overline{T}^2_n M/G^2_n\,. \tag{9.38}$$

Consider $Z \in \operatorname{reg} \overline{T}^2_n M$ with coordinates (y^p, y^p_i, y^p_{ij}). Since Z is regular, there exists a section $s\colon \mathbb{R}^n \to \mathbb{R}^n \times M$ such that y^p_i are the additional coordinates of $j^1_x s$. For $(a^i_j, a^i_{jk}) \in G^2_n$ we obtain directly

$$Z \circ a = (y^p, y^p_k a^k_i, y^p_{kl} a^k_i a^l_j + y^p_k a^k_{ij})\,. \tag{9.39}$$

Now we have $\Delta(Z \circ a) = (y^p_{kl} a^k_{[i} a^l_{j]})$. This proves our claim. □

9.7 The reduced torsion and the difference tensor

Consider a submanifold $N \subset M$ of a Cartan space $\mathcal{S}(M)$. In the tangent space $T_{s(x)}E_x$, we have an n-dimensional subspace $\tau^\Gamma_N(x)$ corresponding to $\Gamma^1_N(x)$. The factor space

$$\nu^\Gamma_N(x) = T_{s(x)}E_x/\tau^\Gamma_N(x) \tag{9.40}$$

will be called the vertical space of N at x. Write σ_N for the restriction of σ to Q_N.

Definition. The projection $\widetilde{\sigma}_N$ of $\sigma_N(x)$ into $\nu^\Gamma_N(x)$ is called the reduced torsion of N at x.

By definition, the second order absolute contact differential of N is a semi-holonomic contact 2-element. So Proposition 9.6 implies directly the following geometric result.

Proposition. *The semiholonomic contact* 2*-element determined by* $N \subset M$ *at* $x \in N$ *is holonomic, if and only if the reduced torsion of* N *at* x *vanishes.*

9.8 2-submanifolds in 3-spaces with projective connection

As a concrete example of the ideas in Chapter 9, we discuss a 2-submanifold $N_2 \subset \mathcal{P}_3$ of a 3-space with projective connection. The projective 3-space P_3 is generated by an affine 4-space A_4 and we write $\{u\} \in P_3$ for the point determined by a nonzero vector $u \in A_4$. We fix a basis u_0, u_1, u_2, u_3 of A_4 and define $c = \{u_0\}$ and c_2 as the linear space in T_cP_3 corresponding the 2-plane determined by $\{u_0\}$, $\{u_1\}$, $\{u_2\}$. The Maurer-Cartan form of the projective group $GP(3)$ is (φ^a_b) with $\varphi^a_a = 0$, a, $b = 0, 1, 2, 3$, and we have

$$d\varphi^b_a = \varphi^c_a \wedge \varphi^b_c \quad \text{with} \quad \varphi^a_a = 0\,. \tag{9.41}$$

The differential equations of H are $\varphi^1_0 = \varphi^2_0 = \varphi^3_0 = 0$. One verifies directly that condition (9.8) is satisfied. Then the relation

$$d\varphi^3_0 = \varphi^0_0 \wedge \varphi^3_0 + \varphi^1_0 \wedge \varphi^3_1 + \varphi^2_0 \wedge \varphi^3_2 + \varphi^3_0 \wedge \varphi^3_3 \tag{9.42}$$

implies that the additional differential equations of H are $\pi^3_1 = 0$, $\pi^3_2 = 0$. The restriction (ω^a_b), $\omega^a_a = 0$ of the connection form ω of $\mathcal{P}_3$ to the first order frames of N_2 is characterized by $\omega^3_0 = 0$. If we write $\omega^1_0 = \omega^1$, $\omega^2_0 = \omega^2$, then (9.13) yields

$$\omega^3_1 = b_{11}\omega^1 + b_{12}\omega^2\,, \quad \omega^3_2 = b_{21}\omega^1 + b_{22}\omega^2\,. \tag{9.43}$$

Then (9.42) is of the form

$$0 = \omega^1 \wedge \omega^3_1 + \omega^2 \wedge \omega^3_2 + 2R^3_0\omega^1 \wedge \omega^2\,. \tag{9.44}$$

Hence (9.43) implies $2R^3_0 = b_{21} - b_{12}$. So the equation (9.43) are of the form

$$\varphi^3_1 = x_{11}\varphi^1_0 + x_{12}\varphi^2_0\,, \quad \varphi^3_2 = x_{21}\varphi^1_0 + x_{22}\varphi^2_0\,. \tag{9.45}$$

Applying the procedure from Section (9.4), we obtain the equations of the infinitesimal action of H_1 on $\overline{S}^2_{21}$

$$\begin{aligned}
&dx_{11} + x_{11}(\pi^0_0 - 2\pi^1_1 + \pi^3_3) - x_{12}\pi^2_1 - x_{21}\pi^2_1 = 0\,,\\
&dx_{12} + x_{12}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^1_2 - x_{22}\pi^2_1 = 0\,,\\
&dx_{21} + x_{21}(\pi^0_0 - \pi^1_1 - \pi^2_2 + \pi^3_3) - x_{11}\pi^1_2 - x_{22}\pi^2_1 = 0\,,\\
&dx_{22} + x_{22}(\pi^0_0 - 2\pi^2_2 + \pi^3_3) - x_{12}\pi^1_2 - x_{21}\pi^1_2 = 0\,.
\end{aligned} \tag{9.46}$$

In particular,

$$d(x_{21} - x_{12}) + (x_{21} - x_{12})(\pi_0^0 - \pi_1^1 - \pi_2^2 + \pi_3^3) = 0\,. \tag{9.47}$$

so that R_0^3 is a relative invariant. Further, in the non-parabolic case $x_{12}x_{21} - x_{11}x_{22} \neq 0$ we find

$$d\big((x_{21} - x_{12})^2/(x_{12}x_{21} - x_{11}x_{22})\big) = 0\,. \tag{9.48}$$

Hence $(R_0^3)^2/(b_{12}b_{21} - b_{11}b_{22})$ is an absolute invariant. Its geometric interpretation was deduced already by É. Cartan, [Ca37b].

Remark. Some general aspects of the holonomicity problem for Γ_N^r in the case $r > 2$ are studied in [Ko71a]. The case $N_2 \subset \mathcal{P}_3$ is treated geometrically in the third order in [Ko71b].

9.9 Remarks

Some further results on the universality of the geometric objects for submanifolds are outlined in [KoVi]. In particular, we described there the universal tensor bundles for submanifolds (in Section 7 of [KoVi]) and the universal induced bundles over submanifolds (in Section 9 of [KoVi]).

Chapter 10

Appendix: Product preserving bundle functors on $\mathcal{M}f$

10.1 Bundle functors on $\mathcal{M}f$

A bundle functor F on $\mathcal{M}f$ transforms every manifold M into a fibered manifold $B_M\colon FM \to M$ and every smooth map $f\colon M \to N$ into an $\mathcal{FM}$-morphism

$$\begin{array}{ccc} FM & \xrightarrow{Ff} & FN \\ {\scriptstyle B_M}\downarrow & & \downarrow{\scriptstyle B_N} \\ M & \xrightarrow{f} & N \end{array}$$

Hence the base projections B_M form a natural transformation of F to the identity functor on the bases.

In accordance with Section 2.4, F is said to preserve products, if the following diagram commutes

$$\begin{array}{ccccc} F(M\times N) & = & FM & \times & FN \\ {\scriptstyle B_{M\times N}}\downarrow & & {\scriptstyle B_M}\downarrow & & \downarrow{\scriptstyle B_N} \\ M\times N & = & M & \times & N \end{array}$$

10.2 $F\mathbb{R}$ is a vector space

Let $a\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the addition of reals. Applying F, we obtain

$$\begin{array}{ccc} F\mathbb{R}\times F\mathbb{R} & \xrightarrow{Fa} & F\mathbb{R} \\ {\scriptstyle B_{\mathbb{R}}}\downarrow \;\; {\scriptstyle B_{\mathbb{R}}}\downarrow & & \downarrow{\scriptstyle B_{\mathbb{R}}} \\ \mathbb{R}\times\mathbb{R} & \xrightarrow{a} & \mathbb{R} \end{array} \tag{10.1}$$

Clearly, (10.1) endows $F\mathbb{R}$ with the structure of an Abelian group, if we set

$$Fa(Z_1,Z_2)\in F\mathbb{R}\,,\quad Z_1,Z_2\in F\mathbb{R}\,. \tag{10.2}$$

Let $\mu\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the multiplication of reals. We define an action of $\mathbb{R}$ on $F\mathbb{R}\to\mathbb{R}$ by

$$\mathbb{R}\times F\mathbb{R}\to F\mathbb{R}\,,\quad (c,Z)\mapsto F\mu(\widehat{c},Z)\,,\quad c\in\mathbb{R}\,, \tag{10.3}$$

where $\widehat{c}$ is the constant injection of c. We have

$$c_1(c_2Z)=(c_1c_2)Z\,,\quad c_1,c_2\in\mathbb{R},\ Z\in F\mathbb{R}\,.$$

One verifies directly the following assertion.

Proposition. *$F\mathbb{R}$ with the addition* (10.2) *and the multiplication by reals* (10.3) *is a vector space.*

Since $F\mathbb{R}$ is a classical manifold, even its dimension as a vector space is finite.

10.3 $F\mathbb{R}$ is a Weil algebra

Applying F to $\mu\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, we obtain

$$\begin{array}{ccc} F\mathbb{R}\times F\mathbb{R} & \xrightarrow{F\mu} & F\mathbb{R} \\ {\scriptstyle B_{\mathbb{R}}}\downarrow \;\; {\scriptstyle B_{\mathbb{R}}}\downarrow & & \downarrow{\scriptstyle B_{\mathbb{R}}} \\ \mathbb{R}\times\mathbb{R} & \xrightarrow{\mu} & \mathbb{R} \end{array}$$

One verifies easily that the vector space $F\mathbb{R}$ is an algebra with respect to $F\mu$.

Write $F^o\mathbb{R}=\{Z\in F\mathbb{R}, Z(o)=0, o\in F\mathbb{R}, 0\in\mathbb{R}\}$. Further, consider the constant injection $\mathbb{R}\hookrightarrow F\mathbb{R}$, $y\mapsto\widehat{y}$. Clearly, $\widehat{1}\in F\mathbb{R}$ is the unit of $F\mathbb{R}$, so that $F\mathbb{R}$ is a unital algebra.

If we restrict F to $\mathcal{M}f_m$, we obtain a bundle functor of a finite order r by Chapter V of [KMS]. Thus, every $Z \in F^o\mathbb{R}$ depends on an r-th order of a map $\mathbb{R} \to \mathbb{R}$, $r \geq 1$. Hence $(Z)^{r+1} = 0$. So, we have proved

Proposition. $\mathbb{R} \times F^o\mathbb{R}$ *is a Weil algebra with the nilpotent part* $F^o\mathbb{R}$.

10.4 F is a Weil functor

Write $F\mathbb{R} = A$, so that $F^o\mathbb{R} = N$ is the nilpotent part of A. Since F preserves products, we have $F\mathbb{R}^m = A^m$. But F satisfies the localization property, so that f can be constructed purely algebraically according to Section 2.5. Using the constructions of Chapter 2, we deduce $F = T^A$.

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